Mathematics of Voting Systems

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Arrow’s Impossibility Theorem

1) No special treatment of particular voters or candidates

2) Transitivity
   • A>B and B>C implies A>C
   • No cycles

3) Monotonicity
   • A voter changing their ballot in a way favoring cannot cause that candidate’s overall ranking to go down.

4) Independence of irrelevant alternatives
   • Overall relative ranking of two candidates depends on only their relative ranking on voter ballots
Why *independence of irrelevant alternatives* matters: 1995 Figure Skating World Championship

- Rankings prior to Michelle Kwan skating:
  - 1\textsuperscript{st} place: Chen Lu (China)
  - 2\textsuperscript{nd} place: Nicole Bobek (USA)
  - 3\textsuperscript{rd} place: Surya Bonaly (France)

- Rankings after judging of Michelle Kwan:
  - 1\textsuperscript{st} place: Chen Lu (China)
  - 2\textsuperscript{nd} place: Surya Bonaly (France)
  - 3\textsuperscript{rd} place: Nicole Bobek (USA)
  - 4\textsuperscript{th} place: Michelle Kwan (USA)
Plurality: whoever gets the most votes wins

**Strengths**
- Simple ballot to fill out
- Transparent results
- Easy to understand
- Monotonic

**Weaknesses**
- Vote splitting
- Spoilers
- Tactical voting
- Negative campaigning

**1860 US Presidential Election**
- Abraham Lincoln
- Stephen Douglas
- John Breckinridge
- John Bell
Borda count

Point system for field of N candidates, e.g.,

• N-1 points for 1st place
• N-2 points for 2nd place
• ...
• 0 points for last place

(or other point scheme, for instance, weighting 1st place more heavily)

**Strengths**

• Takes into account full set of preferences
• Can promote compromise candidates
• Monotonic

**Weaknesses**

• Vulnerable to strategic voting, such as burying favorite’s main rivals
Borda count: 1999 baseball MVP elections

**AL MVP Voting**

<table>
<thead>
<tr>
<th>Rank</th>
<th>Name</th>
<th>Tm</th>
<th>Vote Pts</th>
<th>1st Place</th>
<th>Share</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Ivan Rodriguez</td>
<td>TEX</td>
<td>252.0</td>
<td>7.0</td>
<td>64%</td>
</tr>
<tr>
<td>2</td>
<td>Pedro Martinez</td>
<td>BOS</td>
<td>239.0</td>
<td>8.0</td>
<td>61%</td>
</tr>
<tr>
<td>3</td>
<td>Roberto Alomar</td>
<td>CLE</td>
<td>226.0</td>
<td>4.0</td>
<td>58%</td>
</tr>
<tr>
<td>3</td>
<td>Manny Ramirez</td>
<td>CLE</td>
<td>226.0</td>
<td>4.0</td>
<td>58%</td>
</tr>
<tr>
<td>5</td>
<td>Rafael Palmeiro</td>
<td>TEX</td>
<td>193.0</td>
<td>4.0</td>
<td>49%</td>
</tr>
<tr>
<td>6</td>
<td>Derek Jeter</td>
<td>NYY</td>
<td>177.0</td>
<td>1.0</td>
<td>45%</td>
</tr>
<tr>
<td>7</td>
<td>Nomar Garciaparra</td>
<td>BOS</td>
<td>137.0</td>
<td>0.0</td>
<td>35%</td>
</tr>
<tr>
<td>8</td>
<td>Jason Giambi</td>
<td>OAK</td>
<td>49.0</td>
<td>0.0</td>
<td>12%</td>
</tr>
<tr>
<td>9</td>
<td>Shawn Green</td>
<td>TOR</td>
<td>44.0</td>
<td>0.0</td>
<td>11%</td>
</tr>
<tr>
<td>10</td>
<td>Ken Griffey</td>
<td>SEA</td>
<td>42.0</td>
<td>0.0</td>
<td>11%</td>
</tr>
</tbody>
</table>

28 voters
14 points for 1st place
9 points for 2nd place
8 points for 3rd place
7 points for 4th place
...

Approval voting

• Vote for all candidates you find acceptable
• May reduce vote splitting and support third parties
• Not as expressive as ranked methods

Saari’s example:
• 9,999 voters strongly support A, find B marginally acceptable, and strongly oppose C
• 1 voter strongly supports C, finds B marginally acceptable, and strongly opposes A
Pairwise comparison/Condorcet method

- Winner based on head-to-head matches of all possible pairings of candidates

- Beatpath/CSSD takes into account margins of victory using a weighted directed graph calculation

- Condorcet winner: candidate who wins all head-to-head matches

- Condorcet winner criterion: when a Condorcet winner exists, that candidate should win the election.
Instant runoff voting (IRV)/ranked choice

- Eliminate candidate with least 1st place votes
- Move up candidates and repeat until single winner left

- Burlington, VT 2009 mayoral race used IRV
  - IRV winner was Kiss, followed by Wright then Montroll
  - Montroll was Condorcet winner
  - If Kiss had won more 1st place votes, he would have lost

⇒ IRV is not monotonic
⇒ IRV doesn’t satisfy Condorcet winner criterion
Gibbard-Satterthwaite Theorem

• Tactical voting: dishonest voting to improve ranking of your preferred candidate.

• All ranked voting systems with no special treatment of particular voters or candidates are susceptible to tactical voting.
Gerrymandering

- Incumbent (sweetheart)
  - Ruled OK by court
- Racial
  - Voting Rights Act of 1965
- Partisan
  - No clear measure

- Packing and cracking

http://www.redistrictingthenation.com
Baker vs Carr, 1962 Supreme Court case

- “One person, one vote”
  - Each individual is weighted equally in apportionment (doesn’t matter whether legally able to vote or not)
- Established right of federal courts to review redistricting maps (redrawn every 10 years after census)
- Found Tennessee district map unconstitutional
- Districts did not reflect movement of population to cities
- 2/3 of representatives elected by 1/3 of the state population
Cooper vs Harris: North Carolina district map

• Supreme Court ruled 5-3 earlier this week that Districts 1 and 12 exhibit unconstitutional racial gerrymandering
  • District 12 elected African-American-favored candidates with 64-72% of vote
  • New map increased packing of African-American voters

2003-13 map: 7 Dem to 6 Rep seats in 2011
2013-16 map: 10 Rep to 3 Dem seats in 2015
Quantifying partisan gerrymandering

Efficiency gap

- Stephanopoulos and McGhee
- Assesses “wasted votes” in 2-party election
  - If a party loses the election, all of that party’s votes are wasted.
  - If a party wins the election, the votes past 50% are wasted.
- Sum wasted votes for each party across the districts in that state
- Find difference in total wasted votes between the 2 parties, divided by total # of votes

<table>
<thead>
<tr>
<th>Red</th>
<th>Blue</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>1 vs 4 wasted</td>
<td>Efficiency gap of 30%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Red</th>
<th>Blue</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>1 vs 4 wasted</td>
<td></td>
</tr>
</tbody>
</table>
Quantifying partisan gerrymandering

Efficiency gap of zero doesn’t imply proportional representation

<table>
<thead>
<tr>
<th>District</th>
<th>Red</th>
<th>Blue</th>
<th>Winner</th>
<th>Wasted votes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>4</td>
<td>Red</td>
<td>1 vs 4</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>4</td>
<td>Red</td>
<td>1 vs 4</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>4</td>
<td>Red</td>
<td>1 vs 4</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>6</td>
<td>Blue</td>
<td>4 vs 1</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>7</td>
<td>Blue</td>
<td>3 vs 2</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>7</td>
<td>Blue</td>
<td>3 vs 2</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>7</td>
<td>Blue</td>
<td>3 vs 2</td>
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<tr>
<td>8</td>
<td>3</td>
<td>7</td>
<td>Blue</td>
<td>3 vs 2</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>7</td>
<td>Blue</td>
<td>3 vs 2</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>7</td>
<td>Blue</td>
<td>3 vs 2</td>
</tr>
<tr>
<td>Total</td>
<td>40</td>
<td>60</td>
<td></td>
<td>25 vs 25</td>
</tr>
</tbody>
</table>

- 100 voters in 10 districts
- 40 total Red voters
- 60 total Blue voters
- Red wins 3 districts
- Blue wins 7 districts
- Efficiency gap = 0
- Biased toward Blue
Felony disenfranchisement in the US

• Depends on state laws
• Overall in US, 7.7% of black adults disenfranchised, compared to 1.8% of non-black adults.
• Large prison populations also used as form of gerrymandering (count as population but can’t vote)
• States with most severe laws:
  • Florida (21% of African-Americans disenfranchised)
  • Kentucky (26%)
  • Virginia (22%)
  • Up to 40% of black men disenfranchised in these states

http://politicalmaps.org/6-million-lost-voters-state-level-estimates-of-felony-disenfranchisement-2016/
Thank you for listening!