Mathematics of Voting Systems

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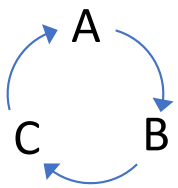
Arrow's Impossibility Theorem

1) No special treatment of particular voters or candidates

2) Transitivity
   - A>B and B>C implies A>C
   - No cycles



3) Monotonicity
   - A voter changing their ballot in a way favoring cannot cause that candidate's overall ranking to go down.

4) Independence of irrelevant alternatives
   - Overall relative ranking of two candidates depends on only their relative ranking on voter ballots

Why *independence of irrelevant alternatives* matters: 1995 Figure Skating World Championship

- Rankings prior to Michelle Kwan skating:
  - 1st place: Chen Lu (China)
  - 2nd place: Nicole Bobek (USA)
  - 3rd place: Surya Bonaly (France)

- Rankings after judging of Michelle Kwan:
  - 1st place: Chen Lu (China)
  - 2nd place: Surya Bonaly (France)
  - 3rd place: Nicole Bobek (USA)
  - 4th place: Michelle Kwan (USA)

Plurality: whoever gets the most votes wins

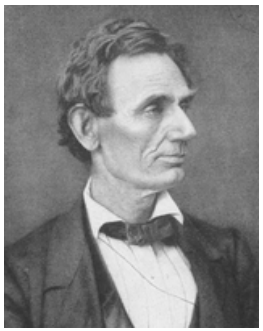
Strengths

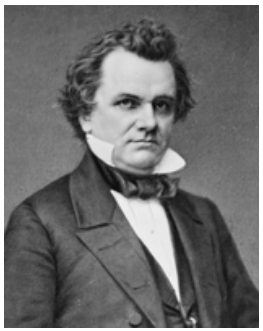
- Simple ballot to fill out
- Transparent results
- Easy to understand
- Monotonic

Weaknesses

- Vote splitting
- Spoilers
- Tactical voting
- Negative campaigning

- 1860 US Presidential Election
  - Abraham Lincoln
  - Stephen Douglas
  - John Breckinridge
  - John Bell





Borda count

Point system for field of N candidates, e.g.,

- N-1 points for 1st place
- N-2 points for 2nd place
- …
- 0 points for last place

(or other point scheme, for instance, weighting 1st place more heavily)

Strengths

- Takes into account full set of preferences
- Can promote compromise candidates
- Monotonic

Weaknesses

- Vulnerable to strategic voting, such as burying favorite's main rivals

Borda count: 1999 baseball MVP elections

AL MVP Voting Share & more ▼ Glossary

			Voting Results		
Rank	**Name**	**Tm**	**Vote Pts**	**1st Place**	**Share**
1	Ivan Rodriguez	TEX	252.0	7.0	64%
2	Pedro Martinez	BOS	239.0	8.0	61%
3	Roberto Alomar	CLE	226.0	4.0	58%
3	Manny Ramirez	CLE	226.0	4.0	58%
5	Rafael Palmeiro	TEX	193.0	4.0	49%
6	Derek Jeter	NYY	177.0	1.0	45%
7	Nomar Garciaparra	BOS	137.0	0.0	35%
8	Jason Giambi	OAK	49.0	0.0	12%
9	Shawn Green	TOR	44.0	0.0	11%
10	Ken Griffey	SEA	42.0	0.0	11%

28 voters

14 points for 1st place
9 points for 2nd place
8 points for 3rd place
7 points for 4th place
…

http://www.baseball-reference.com/awards/awards_1999.shtml

Approval voting

- Vote for all candidates you find acceptable
- May reduce vote splitting and support third parties
- Not as expressive as ranked methods

Sample Approval Voting Ballot (2000 US Election)		
Presidential Candidate	→	**Use an "X" to select as many candidates as you wish.**
1: Pat Buchanan	→	
2: George W. Bush	→	
3: Al Gore	→	
4: Ralph Nader	→	

Saari's example:

- 9,999 voters strongly support A, find B marginally acceptable, and strongly oppose C
- 1 voter strongly supports C, finds B marginally acceptable, and strongly opposes A

Pairwise comparison/Condorcet method

- Winner based on head-to-head matches of all possible pairings of candidates

- Beatpath/CSSD takes into account margins of victory using a weighted directed graph calculation

- *Condorcet winner*: candidate who wins all head-to-head matches

- *Condorcet winner criterion*: when a Condorcet winner exists, that candidate should win the election.

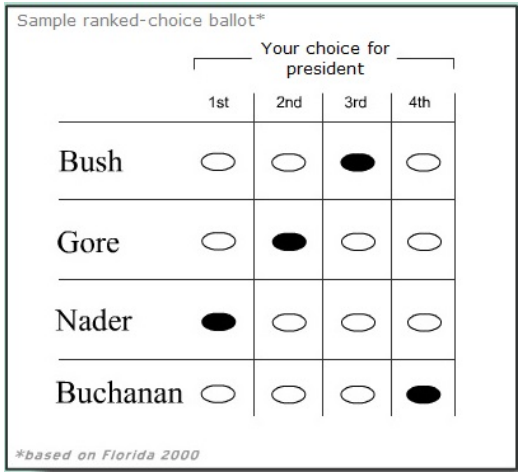
Instant runoff voting (IRV)/ranked choice

- Eliminate candidate with least 1st place votes
- Move up candidates and repeat until single winner left

- Burlington, VT 2009 mayoral race used IRV
  - IRV winner was Kiss, followed by Wright then Montroll
  - Montroll was Condorcet winner
  - If Kiss had won more 1st place votes, he would have lost

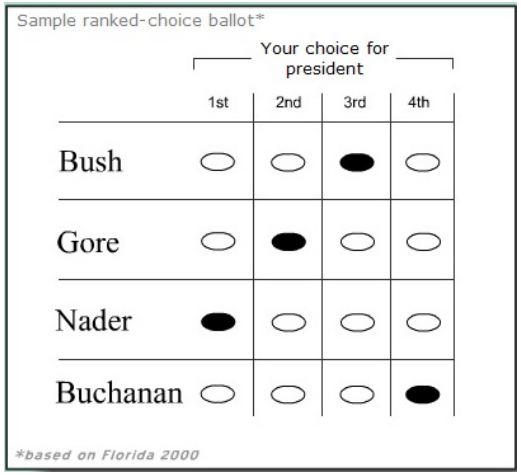
  ➡ IRV is not monotonic

  ➡ IRV doesn't satisfy Condorcet winner criterion



Gibbard-Satterthwaite Theorem

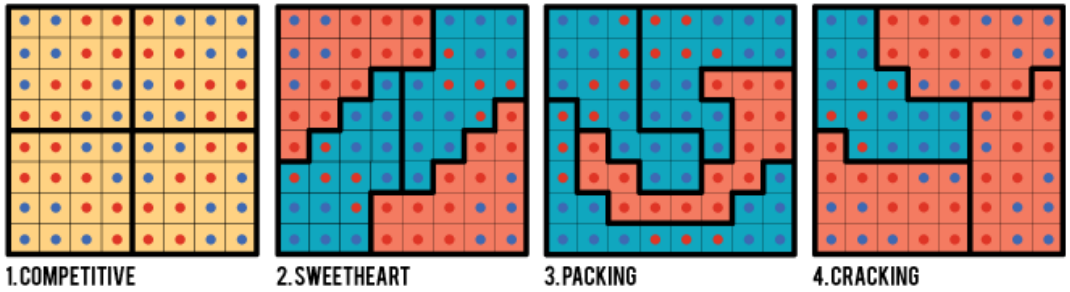
- Tactical voting: dishonest voting to improve ranking of your preferred candidate.
- All ranked voting systems with no special treatment of particular voters or candidates are susceptible to tactical voting.



Gerrymandering

- Incumbent (sweetheart)
  - Ruled OK by court
- Racial
  - Voting Rights Act of 1965
- Partisan
  - No clear measure

- Packing and cracking

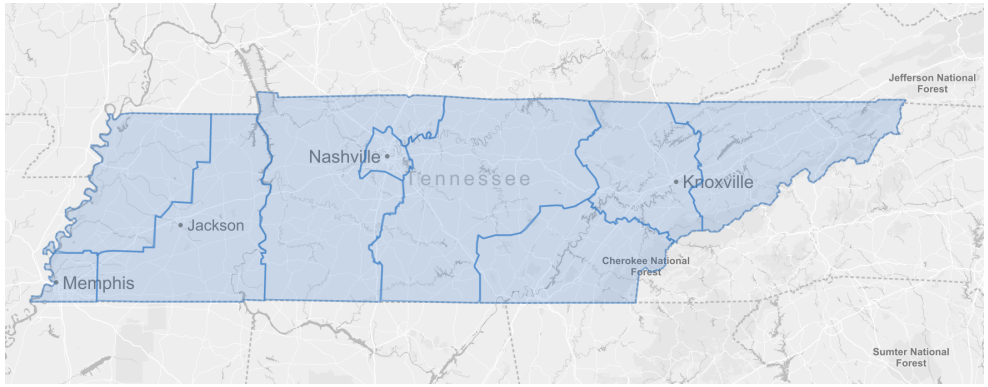


http://www.redistrictingthenation.com

Baker vs Carr, 1962 Supreme Court case

- “One person, one vote”
  - Each individual is weighted equally in apportionment (doesn’t matter whether legally able to vote or not)
- Established right of federal courts to review redistricting maps (redrawn every 10 years after census)
- Found Tennessee district map unconstitutional

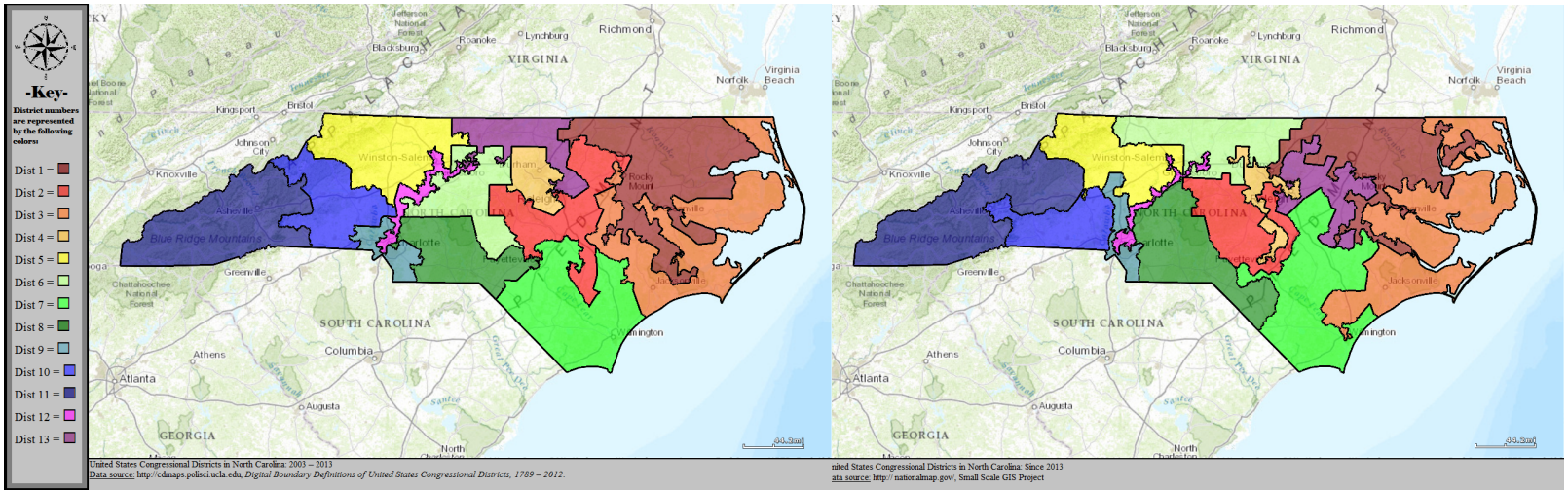
github.com/JeffreyBLewis/congressional-district-boundaries



- Districts did not reflect movement of population to cities
- 2/3 of representatives elected by 1/3 of the state population

Cooper vs Harris: North Carolina district map

- Supreme Court ruled 5-3 earlier this week that Districts 1 and 12 exhibit unconstitutional racial gerrymandering
  - District 12 elected African-American-favored candidates with 64-72% of vote
  - New map increased packing of African-American voters



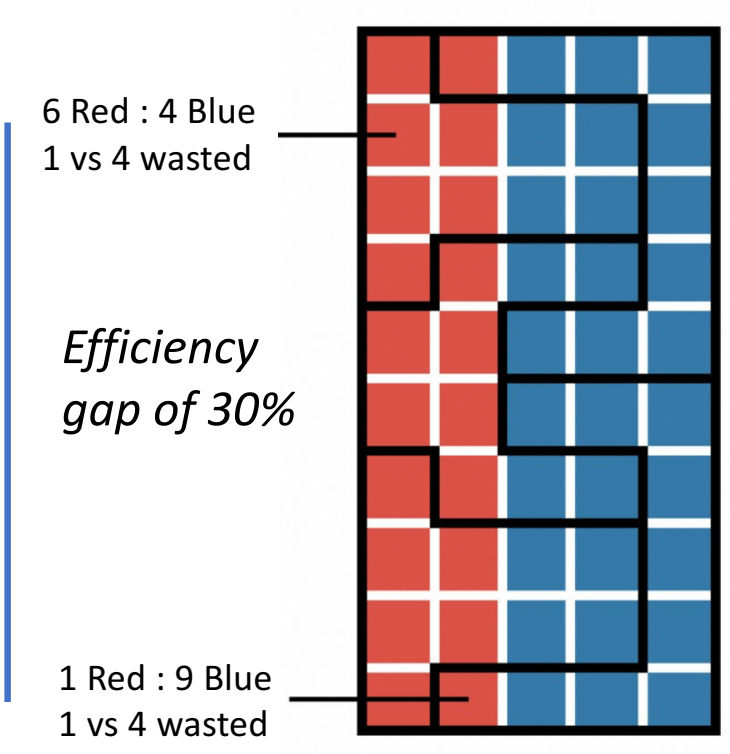
2003-13 map: 7 Dem to 6 Rep seats in 2011

2013-16 map: 10 Rep to 3 Dem seats in 2015

Quantifying partisan gerrymandering

Efficiency gap

- Stephanopoulos and McGhee
- Assesses "wasted votes" in 2-party election
  - If a party loses the election, all of that party's votes are wasted.
  - If a party wins the election, the votes past 50% are wasted.
- Sum wasted votes for each party across the districts in that state
- Find difference in total wasted votes between the 2 parties, divided by total # of votes



Quantifying partisan gerrymandering

Efficiency gap of zero doesn't imply proportional representation

District	Red	Blue	Winner	Wasted votes
1	6	4	Red	1 vs 4
2	6	4	Red	1 vs 4
3	6	4	Red	1 vs 4
4	4	6	Blue	4 vs 1
5	3	7	Blue	3 vs 2
6	3	7	Blue	3 vs 2
7	3	7	Blue	3 vs 2
8	3	7	Blue	3 vs 2
9	3	7	Blue	3 vs 2
10	3	7	Blue	3 vs 2
Total	40	60		25 vs 25

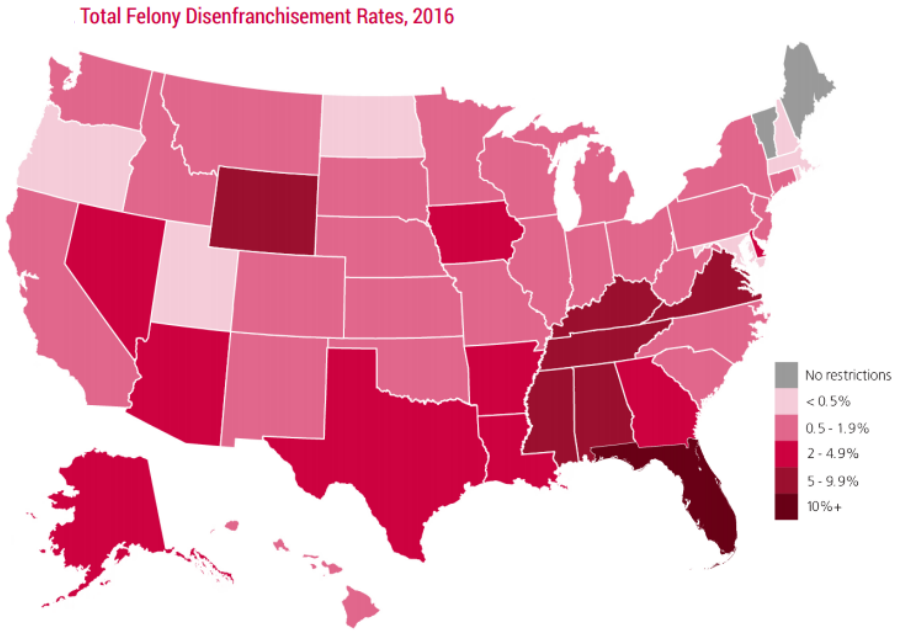
- 100 voters in 10 districts
- 40 total Red voters
- 60 total Blue voters

- Red wins 3 districts
- Blue wins 7 districts

- Efficiency gap = 0
- Biased toward Blue

Felony disenfranchisement in the US

- Depends on state laws
- Overall in US, 7.7% of black adults disenfranchised, compared to 1.8% of non-black adults.
- Large prison populations also used as form of gerrymandering (count as population but can't vote)
- States with most severe laws:
  - Florida (21% of African-Americans disenfranchised)
  - Kentucky (26%)
  - Virginia (22%)
  - Up to 40% of black men disenfranchised in these states



http://politicalmaps.org/6-million-lost-voters-state-level-estimates-of-felony-disenfranchisement-2016/

Thank you for listening!