Final Examination

Mathematics 13 — No books, notes, or electronic devices allowed — December 20, 2004

Please express yourself clearly and show your reasoning. Thank you.

(9) 1. Find an equation for the plane containing the y-axis which is perpendicular to the plane $ax + by + cz = d$.

(9) 2. Prove that if a particle is traveling at a constant speed, then its velocity vector is always perpendicular to its acceleration vector.

(12) 3. Define the function $f : \mathbf{R}^2 \to \mathbf{R}$ by $f(x,y) = \begin{cases} y \ln(x^2 + y^2) & \textit{if} \quad (x,y) \neq (0,0) \\ 0 & \textit{if} \quad (x,y) = (0,0) \end{cases}$

i) Compute the partial derivative $\dfrac{\partial f}{\partial x}(x,y)$ if $(x,y) \neq (0,0)$.

ii) Compute $\dfrac{\partial f}{\partial x}(0,0)$ using the limit definition of partial derivative.

iii) Is $\dfrac{\partial f}{\partial x}$ continuous at $(0, 0)$? Either explain why it is or show why it is not.

(8) 4. Let $T = F(u, v, w)$ be a differentiable function of three variables.

Suppose $u = x - y$, $v = 2y - 2x$, and $w = 3x - 3y$.

Show that $\dfrac{\partial T}{\partial x} + \dfrac{\partial T}{\partial y} = 0$.

(8) 5. Let the relation $ye^z + xz - x^2 - y^2 = 0$ determine a surface. Evaluate the partial derivatives $\dfrac{\partial z}{\partial x}$ *and* $\dfrac{\partial z}{\partial y}$ at the point $(0, 1, 0)$ on this surface.

(10) 6. Let $h(x, y) = -x^3 + 4xy - 2y^2 + 1$. Find the locations of any relative maxima, relative minima, and saddle points for h. Classify each point, and justify your conclusions.

(12) 7. Let $f(x, y)$ represent the temperature at the location (x, y) on a rectangular computer chip, having corners at (0, 0), (50, 0), (50, 25), and (0, 25). Let f have continuous partial derivatives everywhere and suppose that at (x, y) = (5, 10): $f = 300$, $\frac{\partial f}{\partial x} = 30$, and $\frac{\partial f}{\partial y} = -40$.

i) Find a vector $a\vec{\mathbf{i}} + b\vec{\mathbf{j}}$ pointing from (5, 10) in the direction of greatest rate of decrease of f.

ii) Find the rate of change of f at (5, 10) in the direction for which it decreases most rapidly.

iii) Use differentials to estimate the value of f at the location .1 units to the right of (5, 10) and .2 units above (5, 10). (Note: The positive x-direction is to the right, the positive y-direction is upward.)

(10) 8. Let C be the positively oriented (i.e. counterclockwise) boundary of the region in the first quadrant which is enclosed by the graphs of $y = x$ and $y = x^3$. Evaluate $\int_C x^2 y\,dx + \sin(y^2)\,dy$.

(10) 9. Consider the line integral $\int_C (Kxy^3 - ye^{xy})dx + (3x^2y^2 - xe^{xy})dy$, where K represents a positive constant.

i) Evaluate this line integral if C is the line segment from the origin to (1, 2).

ii) If there is a value of K for which this line integral is independent of path, find such a K and determine the corresponding potential function.

(12) 10. Let S be the region enclosed by the plane $z = 3$ and the cone $z = \sqrt{\frac{x^2}{3} + \frac{y^2}{3}}$.

i) Using cylindrical coordinates, determine the volume of S.

ii) Using spherical coordinates, determine the volume of S.

iii) Using surface integration, find the surface area of the *conical* portion of the surface of S, i.e. the portion for $z < 3$.

Math 13 Final Exam Fall 2004 Solutions: Please let me know if you find any errors!

1. Plane contains the point (0,0,0) and the vectors <0,1,0> and <a,b,c>. Hence the normal vector to the plane is the cross product of <0,1,0> and <a,b,c> and the equation for the plane is cx=az.

2. Speed is constant, so **v**·**v**=C. Take a derivative: 2**v**·**v**′=0. So the velocity vector is orthogonal to the acceleration vector.

3. (i) $2xy/(x^2+y^2)$

   (ii) $\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{0-0}{h} = 0$

   (iii) Not continuous since the limit along $y=x$ is 1 as approach (0,0).

4. Chain rule: $\frac{\partial T}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial x}$ and similarly for y. Simplify sum of partials of T to see that the three partials of F cancel out.

5. 0 and 1.

6. Local maximum at (4/3,4/3) and saddle at (0,0).

7. (i) <-3/5,4/5>; (ii) -50; (iii) $f(5.1,10.2)$ is approximately 295.

8. -1/12

9. (i) $(8K+24)/5-e^2+1$

   (ii) $K=2$ and $f(x,y)=x^2y^3-e^{xy}$.

10. (i) $0 \le \theta \le 2\pi, 0 \le r \le 3\sqrt{3}, r/\sqrt{3} \le z \le 3$ yields 27π;

    (ii) $0 \le \theta \le 2\pi, 0 \le \phi \le \pi/3, 0 \le \rho \le 3/\cos\phi$ also yields 27π;

    (iii) $18\sqrt{3}\pi$ is the surface area of the lateral surface of the cone (excluding the top).