Final Examination

Mathematics 13 Section 2 December 19, 2007

No books, notes, calculators or communications devices allowed.

Please express yourself clearly, and show your reasoning. Thank you.

(10) 1. Consider the plane $x + 2y + 2z = 0$ and the point $(2,2,1)$.

i) Give the equations of a line L through $(2,2,1)$ and parallel to the given plane.

ii) Find the distance between L and the given plane.

2. Consider motion along the curve C defined by $x = 2t + 3, \quad y = t^2 - 1$.

(2) i) Find the velocity and the acceleration vectors for this motion.

(8) ii) Find the following four scalar quantities associated with this curve:

Speed, curvature, and the tangential and the normal components of acceleration.

(8) 3. Let $w = F(xz, yz)$. Show that $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = z\frac{\partial w}{\partial z}$.

Hint: Let $u = xz,\ v = yz$ and compute the $x, y,$ and z derivatives of $w = F(u,v)$.

(12) 4. For $(x,y) \neq (0,0)$, let $f(x,y) = \frac{(y-2x)^4}{(x^2+y^2)^2}$ *and* $g(x,y) = \frac{(y-2x)^5}{(x^2+y^2)^2}$.

Also let $f(0,0) = g(0,0) = 0$.

i) Which, if any, of these functions *fail* to have a limit as $(x,y) \to (0,0)$?

Demonstrate the failure in any such case.

ii) Which, if any, of these functions have a partial derivative with respect to x at $(0,0)$?

Compute the partial derivative in any such case.

(10) 5. Let $f(x,y) = x^4/4 + 4xy + y^4/4$.

i) Locate and classify (as local maxima, local minima, or saddle points) any critical points of f.

ii) Suppose the domain of f is now restricted to the square S: $\{-10 \le x \le 10,\ -10 \le y \le 10\}$. Find the maximum and the minimum of f on S and give the x, y coordinates of the points where these extreme values occur.

(9) 6. The elevation (in miles) on a mountain is given by $z = f(x,y) = x^2 + xy + y^2$.

According to your map, you are located over $x = -1,\ y = 1$.

i) If you become ill and wish to *descend* most rapidly, in which direction do you move?

ii) Find the maximal rate of descent, as a number. I.e. Determine the directional derivative in direction found in i).

iii) If you just want to take it easy and go in a direction that neither increases nor decreases your elevation, what are the possible directions, expressed as unit vector(s)?

(10) 7. Consider the integral $\iint_D (x^2 + y^2)^{\frac{3}{2}}\, dx\, dy$.

i) Evaluate this integral if D is the disc of radius 2 centered at the origin.

ii) Evaluate this integral if D is the disc of radius 1 centered at $(0,1)$.

(8) 8. Determine the volume of the solid S lying on or above the x-y plane and enclosed by the surface $\rho = \cos\phi$, by evaluating a triple integral taken over S.

(12) 9. i) Let C be the contour $y = \frac{x^2}{\pi}$ starting at (0,0) and ending at (π, π) Evaluate the integral:

$$\int_C (\sin y - y \sin x)dx + (\cos x + x \cos y)dy$$

ii) Verify Green's Theorem for the vector field $M(x,y)\vec{i} + N(x,y)\vec{j} = xy\vec{i} + y^2\vec{j}$

on the region bounded by $y = x^2$ and $y = x$. That is, compute the line integral and the double integral as given in Green's Theorem, and show that they agree.

Solutions to Math 13 Final Exam from Fall 2007

Please let me know if you find any errors in these answers. Thank you!

1. (i) There are many possible answers. Choose any vector <*a,b,c*> such that the dot product with <1,2,2> equals zero, for example, <2,0,-1>. Let x=2+at, y=2+bt, z=1+ct. (ii) We need to find the distance between the plane and the point (2,2,1). The origin lies on the plane, so <2,2,1> is a vector pointing from the plane to the point (2,2,1). The distance equals the absolute value of the scalar projection of <2,2,1> onto the unit normal to the plane: d=|<2,2,1>.<1,2,2>/3|=8/3.

2. (i) **v**(t)=<2,2t> and **a**(t)=<0,2>. (ii) Curvature is $\frac{1}{2}\left(1+t^2\right)^{-3/2}$, $a_T = \frac{2t}{\sqrt{1+t^2}}$, and $a_N = \frac{2}{\sqrt{1+t^2}}$.

3. Substitute $\frac{\partial w}{\partial x} = z\frac{\partial F}{\partial u}$, $\frac{\partial w}{\partial y} = z\frac{\partial F}{\partial v}$, and $\frac{\partial w}{\partial z} = x\frac{\partial F}{\partial u} + y\frac{\partial F}{\partial v}$ into the equation and verify that both sides are the same.

4. (i) $f(x,y)$ fails to have a limit since limit is 16 along x-axis and 1 along y-axis. $g(x,y)$ has limit equal to 0 using polar coordinates. (ii) The partial of f does not exist (limit of difference quotient does not converge). The x-partial of g at (0,0) is -32.

5. (i) (0,0) is a saddle point while (2,-2) and (-2,2) are local minima. (ii) Absolute maxima at (-10,-10) and (10,10) and absolute minimum at (2,-2) and (-2,2).

6. (i) Unit vector opposite from gradient, <1/√2,-1/√2>. (ii) -√2. (iii) ±<1/√2,1/√2>.

7. (i) 64π/5 (ii) 512/75

8. π/6

9. -1/12