Math 13 Final Exam: May 10, 2005

You may not use any calculators or other devices or any notes.

1. (10pt) You plan to calculate the volume inside a stretch of pipeline that is about 1 meter in diameter and 1 kilometer long. With which measurement should you be more careful, the length or the diameter? Why?

2. (15pt) A closed rectangular box is to have volume 24 cubic feet. The cost of material is \$6 per square foot for the top and bottom, \$4 per square foot for the front and back, and \$1 per square foot for the other two sides. What dimensions minimize the total cost of the box, and what is the minimum cost?

3. (10pt) Suppose you have a sphere of radius R and you plan to remove a cylindrical core of radius a (with axis of symmetry passing through the center of the sphere). If you want the remaining volume (inside the sphere and outside the cylinder) to be a given value V, find the radius a in terms of R and V.

4. (15pt) Which of the following vector fields are conservative (i.e., the gradient of a potential)? For those that are, find a potential function f such that $\mathbf{F} = \nabla f$. Otherwise, show that they are not conservative fields.
   a. $\mathbf{F} = 3x^2 y\mathbf{i} + x^3\mathbf{j} + 5\mathbf{k}$
   b. $\mathbf{F} = (x+z)\mathbf{i} - (y+z)\mathbf{j} + (x-y)\mathbf{k}$
   c. $\mathbf{F} = 2xy^3\mathbf{i} + x^2z^3\mathbf{j} + 3x^2yz^2\mathbf{k}$

5. (10pt) A surface is described by $z = 3\sqrt{x^2+y^2}$, $0 \le z \le 6$. Describe what this surface looks like and find the tangent plane to the surface at the point (1,0,3). Does this surface have a tangent plane at (0,0,0)?

6. (20pt) Can the function $f(x,y) = \dfrac{x^2 y}{x^2+y^2}$ be defined at (0,0) so that it is continuous at (0,0)? Do its partial derivatives exist at (0,0), and are they continuous at (0,0)?

7. (10pt) Show that the curve $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + \sqrt{t}\mathbf{j} - \frac{1}{4}(t+3)\mathbf{k}$ is normal to the surface $x^2 + y^2 - z = 3$ at their point of intersection.

8. (10pt) Show that the value of $\oint_C xy^2dx + (x^2y + 2x)dy$ around *any* square C (traversed counterclockwise) depends only on the area of the square and not on its location in the plane.

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Brief Solutions

1. $V=\pi(D/2)^2L$ so the approximate error is $dV= \pi DLdD/2+ \pi(D/2)^2dL$, where L=1000m and D=1m. An error in diameter D has a greater effect than an error in the length L.
2. The constraint is xyz=24, while the function to be minimized is cost=12xy+8xz+2yz. The minimum cost is given by dimensions 1ft by 4 ft by 6 ft with total cost $144.
3. Using cylindrical coordinates or other means, one finds that $V=\frac{4\pi}{3}\left(R^2-a^2\right)^{3/2}$ and so $a=\sqrt{R^2-\left(\frac{3V}{4\pi}\right)^{2/3}}$.
4. a. $f(x,y,z)=x^3y+5z$; b. $f(x,y,z)=x^2/2-y^2/2+xz-yz$; c. curl**F** is not the zero vector, so **F** is not a gradient field.
5. The surface is a cone of height 6 with circular top of radius 2. The tangent plane at (1,0,3) is given by z=3x. There is no well-defined tangent plane at the sharp tip (0,0,0).
6. Define $f(x,y)$ to be 0 at (0,0). Can use polar coordinates to show that f is indeed continuous at (0,0). The partial derivatives exist (and equal 0) but are not continuous at (0,0) since their limits do not exist (can get different values as approach (0,0) from different directions).
7. The point of intersection is (1,1,-1) with t=1. The tangent to the curve (given by $\mathbf{r}'(1)$) at this point is <1/2,1/2,-1/4> while the normal to the surface (given by the gradient) is <2,2,-1>. These vectors are parallel, so the curve is normal to the surface.
8. Apply Green's Theorem to find that the line integral equals 2 times the area of the region, and so only depends on the area of the region and not its boundary (doesn't need to be a square).