Math 13, Section 01, Fall 2010

Final Exam, Wednesday, December 22, 2010

Instructions: Do all twelve numbered problems. If you wish, you may also attempt the three optional bonus questions.
Show all work, including scratch work. Little or no credit may be awarded, **even when your answer is correct**, if you fail to follow instructions for a problem or fail to **justify your answer.**
Simplify your answers whenever possible, and **write legibly**.
If you need more space, use the back of any page.
If you have time, check your answers.

WRITE LEGIBLY. NO CALCULATORS.

1. (**10 points**). Find an equation for the plane that contains the point $(0,3,0)$ and the line $\vec{r}(t) = \langle 4-t, 1+2t, 3t \rangle$.

2. (**15 points**). Let $f(x,y) = \begin{cases} \dfrac{25xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 12 & \text{if } (x,y) = (0,0). \end{cases}$

(2a). Compute the directional derivative $D_{\vec{u}} f(0,0)$, where $\vec{u} = \langle 3/5, 4/5 \rangle$.
(2b). Prove that f is not continuous at $(0,0)$.

3. **(20 points)** Find and classify (as local minimum, local maximum, or saddle point) every critical point of the function $f(x,y) = 2x^2 + 8xy + 4y^3 + y^4$.

4. **(15 points)** Find the maximum and minimum values of the function

$$f(x,y,z) = (x-3)^2 + (y-1)^2 + (z+1)^2$$

on the sphere $x^2 + y^2 + z^2 = 11$.

5. **(20 points)** Let D be the region in the plane that lies above the x-axis, inside the circle $x^2 + y^2 = 2x$, and outside the circle $x^2 + y^2 = 1$. Compute $\iint_D y \, dA$.

6. **(20 points)** Let E be the solid lying inside the sphere $x^2 + y^2 + z^2 = 2$ and above the cone $z = \sqrt{x^2 + y^2}$ in the first octant. Compute $\iiint_E x \, dV$.

7. **(20 points)** Find the volume of the solid bounded by the surface $y = x^2$ and the planes $y = z$ and $z = 1$.

8. **(15 points)** Let C be the curve parametrized by $\vec{r}(t) = \langle t, 3t, t^2 \rangle$ for $0 \leq t \leq 2$, and let $f(x,y,z) = x + y$. Compute $\int_C f \, ds$.

9. **(15 points)** Let C be the boundary of the triangle with vertices $(0,0)$, $(2,0)$, and $(0,1)$, oriented counterclockwise, and let $\vec{F}(x,y) = \langle xy + \sin(\pi x^3), x^2 + 5y^2 \rangle$. Compute $\int_C \vec{F} \cdot d\vec{r}$.

10. **(20 points)** Let $\vec{G}(x,y,z) = \langle x^2 - 5yz, xy + z, y^2 - 3xz \rangle$.

(10a). Compute $\operatorname{curl} \vec{G}$.

(10b). Compute $\operatorname{div} \vec{G}$.

(10c). Is $\vec{G}$ equal to the gradient of anything? Why or why not? [If so, you do **not** need to find what it is the gradient of.]

(10d). Is $\vec{G}$ equal to the curl of anything? Why or why not? [If so, you do **not** need to find what it is the curl of.]

(10e). Is $\vec{G}$ equal to the divergence of anything? Why or why not? [If so, you do **not** need to find what it is the divergence of.]

11. **(10 points)** Let C be the curve in the plane parametrized by $\vec{r}(t) = \langle t^2 + 1, t^3 - 1 \rangle$ for $0 \le t \le 2$. Compute $\int_C y\,dx + x\,dy$.

12. **(20 points)** Let $\vec{F}(x,y) = \langle 6x - 3x^2y^2, 4 - 2x^3y \rangle$.

(12a). Show that $\vec{F}$ is conservative by finding a potential function for $\vec{F}$.

(12b). Let C be the curve parametrized by $\vec{r}(t) = \langle t\cos(\pi t/2), t^2\sin(\pi t/2) \rangle$ for $1 \le t \le 2$. Compute $\int_C \vec{F} \cdot d\vec{r}$.

OPTIONAL BONUS A. (2 points) Recall that on the homework, you verified that the vector field $\vec{F} = \langle P, Q \rangle = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ has $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$, but you computed $\int_{C_1} \vec{F} \cdot d\vec{r}$ and worked out that it was not zero, where C_1 is the circle of radius 1 centered at the origin, oriented counterclockwise.

Compute $\int_{C_2} \vec{F} \cdot d\vec{r}$, where C_2 is the limaçon $r = 4 + \sin\theta$, oriented counterclockwise.

OPTIONAL BONUS B. (2 points) Find a vector field $\vec{F}(x,y,z)$ such that $\operatorname{curl}(\vec{F}) = \langle x - yz, y - xz, xy - 2z \rangle$.

OPTIONAL BONUS C. (1 point) A massive oil spill earlier in 2010 occurred when an offshore drilling rig exploded, leaving the oil well open on the bottom of the ocean. What was the name of the drilling rig itself?