Math 13, Section 01, Fall 2010

Solutions to the Final Exam

1. Find an equation for the plane that contains the point $(0,3,0)$ and the line $\vec{r}(t) = \big\langle 4-t, 1+2t, 3t\big\rangle$.
Answer. The vector $\vec{a} = \langle -1,2,3\rangle$ is parallel to the line and hence to the plane. Meanwhile, the point $\vec{r}(0) = (4,1,0)$ is on the plane, so the vector $\vec{b} = \langle 4-0, 1-3, 0-0\rangle = \langle 4,-2,0\rangle$ is also parallel to the plane. Thus, we can take the normal vector to be $\vec{a}\times\vec{b} = \langle 6,12,-6\rangle$; or dividing by 6, let $\vec{n} = \langle 1,2,-1\rangle$. Using the point $(0,3,0)$ as a point on the plane, then, the equation for the plane is $1(x-0)+2(y-3)-1(z-0) = 0$, or equivalently, $x+2y-z = 6$.

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2. Let $f(x,y) = \begin{cases} \dfrac{25xy}{x^2+y^2} & \text{if } (x,y)\neq(0,0), \\ 12 & \text{if } (x,y) = (0,0). \end{cases}$

(2a). Compute the directional derivative $D_{\vec{u}}f(0,0)$, where $\vec{u} = \big\langle 3/5, 4/5\big\rangle$.

(2b). Prove that f is not continuous at $(0,0)$.

Answer. (a). By definition, $D_{\vec{u}}f(0,0) = \lim\limits_{h\to 0} \dfrac{f\Big(\frac{3h}{5},\frac{4h}{5}\Big) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{12h^2}{h^2} - 12}{h} = \lim\limits_{h\to 0}\dfrac{0}{h} = 0.$

(b). If we approach $(0,0)$ along the x-axis, the value of f approaches
$\lim\limits_{x\to 0} f(x,0) = \dfrac{0}{x^2} = 0 \neq 12 = f(0,0)$, and therefore f is not continuous at $(0,0)$.

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3. Find and classify (as local minimum, local maximum, or saddle point) every critical point of the function $f(x,y) = 2x^2+8xy+4y^3+y^4$.
Answer. $f_x(x,y) = 4x+8y$, and $f_y(x,y) = 8x+12y^2+4y^3$.
Setting both to zero, we see that $f_x = 0$ gives $x = -2y$, and therefore substituting into $f_y = 0$ gives $4y^3+12y^2-16y = 0$. This last equation factors as $4y(y-1)(y+4) = 0$, giving $y = 0$, $y = 1$, or $y = -4$. Recalling $x = -2y$, then, we have three critical points: $(0,0)$, $(-2,1)$, and $(8,-4)$.
We have $f_{xx} = 4$, $f_{xy} = f_{yx} = 8$, and $f_{yy} = 12(2y+y^2)$. Thus, the discriminant is $D = 48(2y+y^2)-64 = 16(3y^2+6y-4)$.
At $(0,0)$, we have $D = -64 < 0$, so there is a saddle point at $(0,0)$.
At $(-2,1)$, we have $D = 80 > 0$, and $f_{xx} = 4 > 0$, so there is a local minimum at $(-2,1)$.
At $(8,-4)$, we have $D = 320 > 0$, and $f_{xx} = 4 > 0$, so there is a local minimum at $(8,-4)$.

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4. Find the maximum and minimum values of the function $f(x,y,z) = (x-3)^2+(y-1)^2+(z+1)^2$ on the sphere $x^2+y^2+z^2 = 11$.

Answer. Write $g(x,y,z) = x^2+y^2+z^2$ and use Lagrange Multipliers. We have $\nabla f = \langle 2(x-3), 2(y-1), 2(z+1)\rangle$ and $\nabla g = \langle 2x,2y,2z\rangle$. Thus, $\nabla f = \lambda\nabla g$ gives

$$x-3 = \lambda x, \qquad y-1 = \lambda y, \qquad z+1 = \lambda z,$$

along with the original restriction $x^2+y^2+z^2 = 11$.
The first equation gives $(1-\lambda)x = 3$, which implies that $\lambda\neq 1$, and hence $x = 3/(1-\lambda)$. Similarly, the next two equations give $y = 1/(1-\lambda)$ and $z = -1/(1-\lambda)$.
Thus, $x = 3y$ and $z = -y$. Substituting into the fourth equation gives $11y^2 = 11$, and hence $y = \pm 1$. Thus, there are two points to test: $(3,1,-1)$ and $(-3,-1,1)$.
We have $f(3,1,-1) = 0$ and $f(-3,-1,1) = 36+4+4 = 44$. Thus, the maximum value is 44 and the minimum is 0.

5. Let D be the region in the plane that lies above the x-axis, inside the circle $x^2+y^2=2x$, and outside the circle $x^2+y^2=1$. Compute $\iint_D y\,dA$.

Answer. In polar coordinates, the two circles are $r=2\cos\theta$ and $r=1$. Their intersection point in the first quadrant occurs when $\cos\theta = 1/2$ with $0 \le \theta \le \pi/2$, which means $\theta = \pi/3$. Thus,

$$\iint_D y\,dA = \int_0^{\pi/3}\int_1^{2\cos\theta} r^2\sin\theta\,dr d\theta = \frac{1}{3}\int_0^{\pi/3} r^3\sin\theta\Big|_{r=1}^{2\cos\theta} d\theta = \frac{1}{3}\int_0^{\pi/3} 8\cos^3\theta\sin\theta - \sin\theta\,d\theta$$

$$[u=\cos\theta, du=-\sin\theta\,d\theta] \quad = -\frac{1}{3}\int_1^{1/2} 8u^3 - 1\,du$$

$$= \frac{1}{3}(u-2u^4)\Big|_1^{1/2} = \frac{1}{3}\Big(\frac{1}{2}-\frac{1}{8}\Big) - \frac{1}{3}(1-2) = \frac{1}{8}+\frac{1}{3} = \frac{11}{24}.$$

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6. Let E be the solid lying inside the sphere $x^2+y^2+z^2=2$ and above the cone $z=\sqrt{x^2+y^2}$ in the first octant. Compute $\iiint_E x\,dV$.

Answer. In spherical coordinates, $\displaystyle\iiint_E x\,dV = \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{2}} \rho^3\sin^2\phi\cos\theta\,d\rho d\phi d\theta$

$$= \int_0^{\pi/2}\cos\theta\,d\theta\int_0^{\pi/4}\frac{1}{2}-\frac{1}{2}\cos 2\phi\,d\phi\int_0^{\sqrt{2}}\rho^3\,d\rho = \Big[\sin\theta\Big]_0^{\pi/2}\Big[\frac{\phi}{2}-\frac{1}{4}\sin 2\phi\Big]_0^{\pi/4}\Big[\frac{\rho^4}{4}\Big]_0^{\sqrt{2}}$$

$$= (1-0)\cdot\Big(\frac{\pi}{8}-\frac{1}{4}-0+0\Big)\cdot(1-0) = \frac{\pi}{8}-\frac{1}{4}.$$

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7. Find the volume of the solid bounded by the surface $y=x^2$ and the planes $y=z$ and $z=1$.

Answer. The planes intersect along the line $y=z=1$, and the solid lies below $z=1$, above $z=y$, and to the right of $y=x^2$. Projecting the solid onto the xy-plane gives the region bounded by the curve $y=x^2$ and the line $y=1$. These two curves intersect at $(-1,1)$ and $(1,1)$. Thus, the volume is $\displaystyle\int_{-1}^1\int_{x^2}^1\int_y^1 dz\,dy\,dx = \int_{-1}^1\int_{x^2}^1 (1-y)\,dy\,dx = \int_{-1}^1\Big[y-\frac{1}{2}y^2\Big]_{y=x^2}^1 dx$

$$= \int_{-1}^1 \frac{1}{2} - x^2 + \frac{1}{2}x^4\,dx = \frac{x}{2}-\frac{x^3}{3}+\frac{x^5}{10}\Big|_{-1}^1 = 1-\frac{2}{3}+\frac{1}{5} = \frac{1}{3}+\frac{1}{5} = \frac{8}{15}$$

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8. Let C be the curve parametrized by $\vec{r}(t) = \langle t, 3t, t^2\rangle$ for $0\le t\le 2$, and let $f(x,y,z) = x+y$. Compute $\int_C f\,ds$.

Answer. $\vec{r}\,'(t) = \langle 1,3,2t\rangle$, so $\|\vec{r}\,'(t)\| = \sqrt{1+9+4t^2} = \sqrt{10+4t^2}$. Meanwhile, $f(\vec{r}(t)) = 4t$. Thus, $\displaystyle\int_C f\,ds = \int_0^2 4t\sqrt{10+4t^2}\,dt \quad [u=10+4t^2, du=8t\,dt] \quad = \frac{1}{2}\int_{10}^{26}\sqrt{u}\,du = \frac{1}{3}u^{2/3}\Big|_{10}^{26}$

$$= \frac{1}{3}(26^{3/2}-10^{3/2}).$$

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9. Let C be the boundary of the triangle with vertices $(0,0)$, $(2,0)$, and $(0,1)$, oriented counter-clockwise, and let $\vec{F}(x,y) = \langle xy+\sin(\pi x^3), x^2+5y^2\rangle$. Compute $\int_C \vec{F}\cdot d\vec{r}$.

Answer. The 2Dcurl is $\partial_x(x^2+y^2) - \partial_y(xy+\sin(\pi x^3)) = 2x-x = x$, and C bounds a triangle D with hypoteneuse given by $x = 2-2y$. Thus, by Green's Theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D x\,dA = \int_0^1\int_0^{2-2y} x\,dx\,dy = \int_0^1 \frac{1}{2}x^2\Big|_0^{2-2y} dy$$

$$= \int_0^1 2-4y+2y^2\,dy = 2y-2y^2+\frac{2}{3}y^3\Big|_0^1 = 2-2+\frac{2}{3}-0 = \frac{2}{3}$$

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10. Let $\vec{G}(x,y,z) = \langle x^2 - 5yz, xy + z, y^2 - 3xz \rangle$.

(10a). Compute $\operatorname{curl} \vec{G}$.
(10b). Compute $\operatorname{div} \vec{G}$.
(10c). Is $\vec{G}$ equal to the gradient of anything? Why or why not? [If so, you do **not** need to find what it is the gradient of.]
(10d). Is $\vec{G}$ equal to the curl of anything? Why or why not? [If so, you do **not** need to find what it is the curl of.]
(10e). Is $\vec{G}$ equal to the divergence of anything? Why or why not? [If so, you do **not** need to find what it is the divergence of.]

Answer. (a). $\operatorname{curl} \vec{G} = \langle 2y-1, -5y+3z, y+5z \rangle$.
(b). $\operatorname{div} \vec{G} = 2x + x - 3x = 0$.
(c). No, $\operatorname{curl} \vec{G} \neq \vec{0}$ [and $\vec{G}$ is C^1]; therefore, $\vec{G}$ is not the gradient of anything.
(d). Yes, $\operatorname{div} \vec{G} = \vec{0}$ and the domain is all of $\mathbb{R}^3$ (which has no avocado-like holes); therefore, $\vec{G}$ is the curl of something.
(e). No, divergence is a scalar, and $\vec{G}$ is a vector field.

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11. Let C be the curve in the plane parametrized by $\vec{r}(t) = \langle t^2+1, t^3-1 \rangle$ for $0 \le t \le 2$. Compute $\displaystyle\int_C y\,dx + x\,dy$.

Answer. We have $\vec{r}'(t) = \langle 2t, 3t \rangle$, and therefore $\displaystyle\int_C y\,dx + x\,dy = \int_0^2 (t^3-1)(2t) + (t^2+1)(3t^2)\,dt$

$$= \int_0^2 5t^4 + 3t^2 - 2t\,dt = t^5 + t^3 - t^2\Big|_0^2 = 32 + 8 - 4 = 36.$$

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12. Let $\vec{F}(x,y) = \langle 6x - 3x^2y^2, 4 - 2x^3y \rangle$.

(12a). Show that $\vec{F}$ is conservative by finding a potential function for $\vec{F}$.
(12b). Let C be the curve parametrized by $\vec{r}(t) = \langle t\cos(\pi t/2), t^2\sin(\pi t/2) \rangle$ for $1 \le t \le 2$. Compute $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.

Answer. (a). [Note: as suggested by the problem, I'm just going to go ahead and find the potential function, rather than checking the curl first. The only reason to check the curl is to not waste time looking for a potential function that isn't there. But this problem is saying the potential function *is* there, and we are just supposed to find it.]
We need a function $f(x,y)$ such that $f_x = 6x - 3x^2y^2$ and $f_y = 4 - 2x^3y$. Antidifferentiating the first, we get $f = 3x^2 - x^3y^2 + g(y)$ for some function $g(y)$. Thus, $f_y = -2x^3y + g'(y)$, so it suffices to find a function $g(y)$ such that $g'(y) = 4$; the choice $g(y) = 4y$ works. Thus, the function $f(x,y) = 3x^2 + 4y - x^3y^2$ has $\nabla f = \vec{F}$, and thus it is a potential function for $\vec{F}$.

(b). By the Fundamental Theorem of Line Integrals, $\displaystyle\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(2)) - f(\vec{r}(1))$

$$= f(-2,0) - f(0,1) = 12 - 4 = 8.$$

OPTIONAL BONUS A. Recall that on the homework, you verified that the vector field $\vec{F} = \langle P, Q \rangle = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$ has $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$, but you computed $\displaystyle\int_{C_1} \vec{F} \cdot d\vec{r}$ and worked out that it was not zero, where C_1 is the circle of radius 1 centered at the origin, oriented counterclockwise. Compute $\displaystyle\int_{C_2} \vec{F} \cdot d\vec{r}$, where C_2 is the limaçon $r = 4 + \sin\theta$, oriented counterclockwise.

Answer. Let D be the region inside C_2 and outside C_1. By Green's Theorem, $\iint_D Q_x - P_y\, dA = \int_{C_2} \vec{F} \cdot d\vec{r} - \int_{C_1} \vec{F} \cdot d\vec{r}$. (That's because C_2 is the outer boundary, oriented counterclockwise; but C_1 is the inner boundary and so needs to be oriented backwards.) However, since $Q_x - P_y = 0$, we get $\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r}$.
Parametrizing C_1 by $\vec{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \le t \le 2\pi$, we get
$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} -\sin t(-\sin t) + \cos t(\cos t)\, dt = \int_0^{2\pi} dt = 2\pi$.

OPTIONAL BONUS B. Find a vector field $\vec{F}(x,y,z)$ such that $\mathrm{curl}(\vec{F}) = \big\langle x - yz, y - xz, xy - 2z \big\rangle$.

Answer. There are many correct answers. [As soon as you find one solution $\vec{F}$, just add ∇f, for your favorite function $f(x,y,z)$, and you have another solution.] So let's start by trying to rig a choice of $\vec{F} = \langle P, Q, R \rangle$ by looking at the first coordinate and (fairly arbitrarily) trying to get $R_y = x$ and $Q_z = yz$, so that $R_y - Q_z = x - yz$. So we want $R = xy + g(x,z)$ for some g.
Looking at the second coordinate, then, we need $P_z - R_x = y - xz$, which means $P_z = y - xz + R_x = 2y - xz + g_x$. That is, $P = 2yz - xz^2/2 + h(x,y,z)$, where $h_z = g_x$.
Looking at the third coordinate, we need $Q_x - P_y = xy - 2z$, which means $Q_x = x - 2z + P_y = xy + h_y$. However, recall $Q_z = yz$, so that $Q = yz^2/2 + k(x,y)$. Thus, $Q_x = k_x$; so $k_x = xy + h_y$, meaning that $k = x^2y/2 + a(x,y,z)$, where $a_x = h_y$. Note, then, $Q = yz^2/2 + x^2y/2 + a$.
At this point, any choice of the function g, h, a works, provided they fit the only restrictions we have listed: that $h_z = g_x$ and $a_x = h_y$. The easiest way to do that is to pick all three functions to be zero.
Thus, we can choose $\vec{F} = \langle 2yz - xz^2/2, z^2 + x^2y/2, xy \rangle$. A quick check shows that $\mathrm{curl}\,\vec{F} = \langle x - yz, y - xz, xy - 2z \rangle$, as desired.

OPTIONAL BONUS C. A massive oil spill earlier in 2010 occurred when an offshore drilling rig exploded, leaving the oil well open on the bottom of the ocean. What was the name of the drilling rig itself?
Answer. Deepwater Horizon.