Math 211, Section 01, Spring 2012

Solutions to the Final Exam

1. Find an equation for the line of intersection of the planes $2x - y + 5z = 1$ and $x - z = 3$.

Answer. The planes have normals $\vec{n}_1 = \langle 2, -1, 5\rangle$ and $\vec{n}_2 = \langle 1, 0, -1\rangle$, so the line must be parallel to their cross product $\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 5 \\ 1 & 0 & -1 \end{vmatrix} = \langle 1, 5+2, 1\rangle = \langle 1, 7, 1\rangle$. To find a point on the line, plugging the arbitrary choice $z = 0$ into the second plane gives $x = 3$, and so the first plane gives $y = 2x + 5z - 1 = 5$. That is, the point $(3, 5, 0)$ is on both planes and hence the line. So the line is given by $\vec{r}(t) = \langle t + 3, 7t + 5, t\rangle$.
(Note: there are other correct ways to do this problem.)

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2. Let C be the curve in $\mathbb{R}^3$ parametrized by $\vec{r}(t) = \langle \cos t, \sin t, t^2\rangle$, for $0 \le t \le \pi$.

2.a. Write down, **but do not evaluate**, a definite integral giving the arclength of C.

2.b. Compute $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y, z) = \langle yz, -xz, 6z\rangle$.

Answer. (a). We compute $\vec{r}\,'(t) = \langle -\sin t, \cos t, 2t\rangle$, and hence
$\|\vec{r}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 4t^2} = \sqrt{1 + 4t^2}$. So the arclength is $\int_0^\pi \sqrt{1 + 4t^2}\, dt$.

(b). $\int_C \vec{F} \cdot d\vec{r} = \int_0^\pi \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_0^\pi \langle t^2 \sin t, -t^2 \cos t, 6t^2\rangle \cdot \langle -\sin t, \cos t, 2t\rangle\, dt$
$= \int_0^\pi -t^2 \sin^2 t - t^2 \cos^2 t + 12t^3\, dt = \int_0^\pi -t^2 + 12t^3\, dt = -\frac{t^3}{3} + 3t^4 \Big|_0^\pi = 3\pi^4 - \frac{\pi^3}{3}.$

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3. Find the maximum and minimum values of the function $f(x, y) = 4x^2 y$ on the ellipse $4x^2 + y^2 = 36$.

Answer. We use Lagrange Multipliers; let $g(x, y) = 4x^2 + y^2$. Setting $\nabla f = \lambda \nabla g$ gives $8xy = 8\lambda x$ and $4x^2 = 2\lambda y$, along with the original equation $4x^2 + y^2 = 36$.
The first equation rearranges to $x(y - \lambda) = 0$, so that either $x = 0$ or $\lambda = y$.
If $x = 0$, then $g = 36$ gives $y^2 = 36$, and so $y = \pm 6$.
If $\lambda = y$, then the second equation gives $y^2 = 2x^2$, so that $g = 36$ gives $3y^2 = 36$, and hence $y = \pm\sqrt{12} = \pm 2\sqrt{3}$. Solving for x in $x^2 = y^2/2$ gives $x = \pm\sqrt{6}$. (And the two $\pm$'s are independent.)
Thus, we have six points of interest: the two points $(0, \pm 6)$, and the four points $(\pm\sqrt{6}, \pm 2\sqrt{3})$.
Plugging into f gives

$$f(0, \pm 6) = 0, \qquad f(\pm\sqrt{6}, 2\sqrt{3}) = 48\sqrt{3}, \qquad f(\pm\sqrt{6}, -2\sqrt{3}) = -48\sqrt{3}.$$

Hence, the minimum value is $-48\sqrt{3}$, and the maximum is $48\sqrt{3}$.

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4. Find and classify (as local minimum, local maximum, or saddle point) every critical point of the function $f(x, y) = x^2 y - 2x^2 - 6y^2 - 12y$.

Answer. We have $f_x = 2xy - 4x = 2x(y - 2)$, and $f_y = x^2 - 12y - 12$. Solving $f_x = 0$ gives $x = 0$ or $y = 2$.
If $x = 0$, then $f_y = 0$ gives $12y = -12$, and hence $y = -1$. If $y = 2$, then $f_y = 0$ gives $x^2 - 36 = 0$, and hence $x = \pm 6$. Thus, there are three critical points: $(0, -1)$ and $(\pm 6, 2)$.
To test the critical points, we compute $f_{xx} = 2y - 4$, $f_{xy} = 2x$, and $f_{yy} = -12$.
At $(0, -1)$, we have $D = \begin{vmatrix} -6 & 0 \\ 0 & -12 \end{vmatrix} = 72 > 0$, and $f_{xx} = -6 < 0$, so that f has a **local maximum** at $(0, -1)$.

At $(\pm 6, 2)$, we have $D = \begin{vmatrix} 0 & \pm 12 \\ \pm 12 & -12 \end{vmatrix} = -144 < 0$, (note the two $\pm$'s are synchronized), so that f has a **saddle point** at both $(6, 2)$ and $(-6, 2)$.

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5. Let E be the solid bounded by the paraboloid $z = 2x^2 + 2y^2$ and the plane $z = 2$. Compute $\iiint_E z\, dV$.

The paraboloid lies underneath, and the plane above. They intersect when $z = 2$ and $2x^2+2y^2 = 2$, i.e., above the circle $x^2 + y^2 = 1$. The shadow of the solid E on the xy-plane is the disk enclosed by this circle, so we use cylindrical coordinates. The integral is

$$\iiint_E z\, dV = \int_0^{2\pi} \int_0^1 \int_{2r^2}^2 rz\, dz\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^1 \frac{rz^2}{2}\Big|_{z=2r^2}^2 dr = 2\pi \int_0^1 2r - 2r^5\, dr$$
$$= 2\pi \Big[r^2 - \frac{r^6}{3}\Big]_0^1 = 2\pi\Big(1 - \frac{1}{3}\Big) = \frac{4\pi}{3}.$$

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6. Find the volume of the solid bounded by the surfaces $y = 1 - x^2$, $y = 0$, $z = x$, and $z = 2$.

The solid E lies "inside" (i.e., bounded on the right by) the parabolic cylinder $y = 1 - x^2$. It is bounded on the left by $y = 0$ (the xz-plane), below by $z = x$, and above by $z = 2$. When crushed onto the xy-plane, its shadow is the region bounded above by the parabola $y = 1 - x^2$ and below by the x-axis; these two curves intersect at $x = \pm 1$. Thus, the volume is

$$\iiint_E dV = \int_{-1}^1 \int_0^{1-x^2} \int_x^2 dz\, dy\, dx = \int_{-1}^1 \int_0^{1-x^2} 2 - x\, dy\, dx = \int_{-1}^1 (2-x)(1-x^2)\, dx$$
$$= \int_{-1}^1 2 - x - 2x^2 + x^3\, dx = 2x - \frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4}\Big|_{-1}^1$$
$$= \Big(2 - \frac{1}{2} - \frac{2}{3} + \frac{1}{4}\Big) - \Big(-2 - \frac{1}{2} + \frac{2}{3} + \frac{1}{4}\Big) = 4 - \frac{4}{3} = \frac{8}{3}.$$

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7. Let $f(x, y) = \begin{cases} \dfrac{5y^3 - 2xy}{3x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$

7.a. Compute $f_x(0, 0)$ and $f_y(0, 0)$.

7.b. Prove that f is not continuous at $(0, 0)$.

Answer. (a). $f_x(0, 0) = \lim_{h\to 0} \dfrac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \dfrac{0 - 0}{h} = 0$, and

$$f_y(0, 0) = \lim_{h\to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h\to 0} \frac{\frac{5h^3 - 0}{h^2} - 0}{h} = \lim_{h\to 0} \frac{5h^3}{h^3} = \lim_{h\to 0} 5 = 5.$$

(b). Approaching along the line $y = x$, we have

$$\lim_{x\to 0} f(x, x) = \lim_{x\to 0} \frac{5x^3 - 2x^2}{4x^2} = \lim_{x\to 0} \frac{5x - 2}{4} = -\frac{1}{2} \neq 0 = f(0, 0),$$

and therefore f is not continuous at $(0, 0)$.

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8. Let C be the boundary of the triangle in the plane with vertices $(0, 0)$, $(1, 0)$, and $(0, 3)$, oriented **counterclockwise**, and let $\vec{F}(x, y) = \big\langle y^2 + \cos(x^2 - 3), 4xy\big\rangle$.

Compute $\int_C \vec{F} \cdot d\vec{r}$.

Answer. Because it'll be tedious to parametrize the three legs of the triangle separately, and because the integrand looks hard, we use Green's Theorem. We compute

$\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 4y - 2y = 2y$, and since the triangle lies above the interval $[0,1]$ on the x-axis, with hypoteneuse above given by $y = 3 - 3x$, we have $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \int_0^{3-3x} 2y\, dy\, dx$

$$= \int_0^1 y^2 \Big|_{y=0}^{3-3x} dx = \int_0^1 (3-3x)^2\, dx = \int_0^1 9 - 18x + 9x^2\, dx = 9x - 9x^2 + 3x^3 \Big|_0^1 = 9 - 9 + 3 = 3.$$

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9. Let $\vec{F}(x,y) = \langle x - \cos(2y), y^3 + 2x\sin(2y) \rangle$.

9.a. Show that $\vec{F}$ is conservative by finding a potential function for $\vec{F}$.

9.b. Let C be the curve parametrized by $\vec{r}(t) = \langle \sqrt{t^2+9}, e^{t^2-4t} \rangle$ for $0 \le t \le 4$.
Compute $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.

Answer. (a). Solving $\nabla f = \vec{F}$, we have $f_x = x - \cos(2y)$, so that $f(x,y) = \dfrac{x^2}{2} - x\cos(2y) + g(y)$, for some function $g(y)$. But then $y^3 + 2x\sin(2y) = f_y = 2x\sin(2y) + g'(y)$, so that $g'(y) = y^3$. Thus, we may choose g to be any antiderivative of y^3, such as $y^4/4$. That is, $f(x,y) = \dfrac{x^2}{2} + \dfrac{y^4}{4} - x\cos(2y)$ is a potential function for $\vec{F}$; that is, $\nabla f = \vec{F}$.
(b). Since we have a potential function for $\vec{F}$, we use the Fundamental Theorem of Line Integrals. The starting point of C is $\vec{r}(0) = \langle 3, 1\rangle$, and the ending point is $\vec{r}(4) = \langle 5, 1 \rangle$. Thus,
$\displaystyle\int_C \vec{F} \cdot d\vec{r} = f(5,1) - f(3,1) = \Big(\frac{25}{2} - 5\cos 2 + \frac{1}{4}\Big) - \Big(\frac{9}{2} - 3\cos 2 + \frac{1}{4}\Big) = 8 - 2\cos 2.$

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10. Let $f(x,y)$ be a differentiable function, and suppose that:

$f_x(-1,1) = -2$	$f_x(-1,2) = 7$	$f_x(1,1) = 3$	$f_x(1,2) = 4$
$f_y(-1,1) = 2$	$f_y(-1,2) = -1$	$f_y(1,1) = 5$	$f_y(1,2) = -3.$

Let $h(s,t) = f(st - 2t, 3s - t)$. Compute $h_s(1,1)$.

Answer. Write $z = h(s,t) = f(x,y)$, where $x = st - 2t$, and $y = 3s - t$.
By the Chain Rule, we have $h_s(s,t) = \dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s} = f_x(x,y) \cdot t + f_y(x,y) \cdot 3.$
When $(s,t) = (1,1)$, we have $(x,y) = (1-2, 3-1) = (-1,2)$, and so
$h_s(s,t) = f_x(-1,2) \cdot 1 + f_y(-1,2) \cdot 3 = 7 \cdot 1 + (-1) \cdot 3 = 4.$

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11. Let S be the closed surface consisting of the upper half of the sphere $x^2+y^2+z^2 = 4$ with $z \ge 0$, together with the disk $x^2+y^2 \le 4$ in the xy-plane, oriented outward. Let $\vec{G}(x,y,z) = \langle xz, 3yz, x^2y \rangle$. Use the Divergence Theorem to compute the flux $\displaystyle\iint_S \vec{G} \cdot d\vec{S}$ of $\vec{G}$ through S.

Answer. We compute $\operatorname{div}\vec{G} = z + 3z + 0 = 4z$. Let E be the solid hemisphere (of radius 2) enclosed by S. Then $\displaystyle\iint_S \vec{G} \cdot d\vec{S} = \iiint_E 4z\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 (4\rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

$$= \Big(\int_0^{2\pi} d\theta\Big)\Big(\int_0^{\pi/2} \cos\phi\sin\phi\, d\phi\Big)\Big(\int_0^2 4\rho^3\, d\rho\Big) = 2\pi\Big(\frac{1}{2}\sin^2\phi\Big|_0^{\pi/2}\Big)\Big(\rho^4\Big|_0^2\Big) = \pi \cdot 1 \cdot 16 = 16\pi.$$

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12. Let S be the portion of the surface $z = 4 - x^2 - y^2$ in the first octant, and let C be the boundary of S, oriented **clockwise** when viewed from above. (Note that C consists of three arcs, one in each of the three coordinate planes.) Use Stokes' Theorem to compute $\displaystyle\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y,z) = \langle x^5, xy, \sin z \rangle$.

Answer. We compute $\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x^5 & xy & \sin z \end{vmatrix} = \langle 0,0,y\rangle$. We note that the shadow of S on the xy-plane is a quarter-disk, which suggests using a parametrization inspired by cylindrical coordinates. So we use the parametrization $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 4-r^2\rangle$, for $0 \le \theta \le \frac{\pi}{2}$ and $0 \le r \le 2$.
Thus, $\vec{r}_r = \langle \cos\theta, \sin\theta, -2r\rangle$, and $\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$. Their cross product is $\vec{r}_r \times \vec{r}_\theta = \langle —, —, r\rangle$, where I haven't bothered to compute the x- and y-entries, given that the curl I'll be dot-producting with has zeros in those entries. However, the z-entry is positive, which is the wrong direction according to the right-hand rule and the clockwise orientation of C. So we use $\vec{r}_\theta \times \vec{r}_r = \langle —, —, -r\rangle$. Hence, since $y = r\sin\theta$ (from the z-entry of $\operatorname{curl}\vec{F}$), we have

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_0^{\pi/2}\int_0^2 -r^2\sin\theta\, dr\, d\theta = -\Big(\int_0^{\pi/2}\sin\theta\, d\theta\Big)\Big(\int_0^2 r^2\, dr\Big)$$
$$= -\Big(-\cos\theta\Big|_0^{\pi/2}\Big)\Big(\frac{r^3}{3}\Big|_0^2\Big) = -(-0+1)\Big(\frac{8}{3}-0\Big) = -\frac{8}{3}.$$

OPTIONAL BONUS A. Let C be the portion of the graph of $y=\sin x$ from the point $(0,0)$ to the point $(\pi,0)$. Compute $\displaystyle\int_C (9x^2y^2+y)\,dx + (6x^3y - \sin y)\,dy$.

Answer. Write $\vec{F} = \langle P, Q\rangle$, where $P = 9x^2y^2+y$ and $Q = 6x^3y - \sin y$. Let C' be the straight line segment from $(\pi,0)$ to $(0,0)$, i.e., running right-to-left along the x-axis. Together, C and C' enclose a region D, the region under the first hump of $y=\sin x$ above the x-axis. (But note that the orientation of $C+C'$ is negative.)
Parametrizing C' by $\vec{r}(t) = \langle \pi - t, 0\rangle$ for $0\le t\le\pi$, we have $\vec{r}\,'(t) = \langle -1, 0\rangle$, but because $P(x,0) = Q(x,0) = 0$ for all $x\in\mathbb{R}$, we get $\displaystyle\int_{C'}\vec{F}\cdot d\vec{r} = \int_0^\pi \vec{F}(\pi-t,0)\cdot\langle -1,0\rangle\,dt = 0$.
Thus, by Green's Theorem, recalling the negative orientation, we have

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} + \int_{C'}\vec{F}\cdot d\vec{r} = -\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = -\iint_D 18x^2y - 18x^2y - 1\, dA$$
$$= \iint_D 1\, dA = \int_0^\pi\int_0^{\sin x} dy\, dx = \int_0^\pi \sin x\, dx = \cos x\Big|_0^\pi = 1-(-1) = 2.$$

OPTIONAL BONUS B. Let (x_0,y_0) and (x_1,y_1) be two points in the plane for which $x_0, x_1 > 0$ and $x_0^2+y_0^2 = x_1^2+y_1^2 = 1$. Let C_0 be the straight line segment from (x_0,y_0) to $(0,-1)$, and let C_1 be the straight line segment from (x_1,y_1) to $(0,-1)$. Prove that $\displaystyle\int_{C_0}\frac{ds}{\sqrt{y_0-y}} = \int_{C_1}\frac{ds}{\sqrt{y_1-y}}$.

Proof. Fix $i=0$ or $i=1$, and parametrize C_i by $\vec{r}(t) = \langle x_i(1-t), y_i - (y_i+1)t\rangle$, for $0\le t\le 1$. Then $\vec{r}\,'(t) = \langle -x_i, -(y_i+1)\rangle$, and hence

$$\|\vec{r}\,'(t)\| = \sqrt{x_i^2 + (y_i+1)^2} = \sqrt{x_i^2+y_i^2+2y_i+1} = \sqrt{2y_i+2},$$

where we have used the fact that $x_i^2+y_i^2 = 1$. So

$$\int_{C_i}\frac{ds}{\sqrt{y_i-y}} = \int_0^1 \frac{\sqrt{2y_i+2}}{\sqrt{y_i - [y_i-(y_i+1)t]}}\,dt = \int_0^1\frac{\sqrt{2}\sqrt{y_i+1}}{\sqrt{(y_i+1)t}}\,dt = \int_0^1\frac{\sqrt{2}}{\sqrt{t}}\,dt,$$

which, even though it's an improper integral, is clearly independent of i. (In fact, the value is $2\sqrt{2}$.)

OPTIONAL BONUS C. There are five nations that are **permanent** members of the United Nations Security Council. Name them.
Answer. USA, Russia, Britain, France, China. (I.e., the victors in World War II.)