Math 211 Multivariable Calculus Final Exam
Monday December 15, 2014

You have 3 hours for this exam. You may not use books, notes, calculators, cell phones or any other aids. Please turn off all electronic devices, including cell phones.

Explain your answers fully, showing all work for full credit. Answers involving sine, cosine or tangent of angles that are multiples of π, $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$, $\frac{\pi}{6}$ should be evaluated exactly. Follow directions carefully: If the question tells you to use a particular method or definition, then you must use it in order to get credit. The total number of points is 100.

1. (5 points) Find parametric equations for the line passing through the point (1,2,3) that is parallel to both the xy-plane and the plane $x + 2y + 3z = 1$.

2. (5 points) Use the linear approximation of $f(x, y, z) = xy/z$ to estimate $f(2.04, 0.95, 2.02)$.

3. (10 points) Let $f(x, y) = \begin{cases} \dfrac{2x^2 + 3xy + 4y^2}{x^2 + 2y^2} & \text{if } (x, y) \neq (0, 0) \\ 2 & \text{if } (x, y) = (0, 0). \end{cases}$

(a) Compute $f_x(0, 0)$ and $f_y(0, 0)$.

(b) Prove that f is **not** continuous at $(0, 0)$.

4. (10 points) Consider the function $f(x, y) = x^2 + 4y^2$.

(a) Sketch level curves of the function $f(x, y) = x^2 + 4y^2$ for constant values $c = 0$, 4, and 16. On your sketch, draw the gradient vectors at the points $(1, \sqrt{3}/2)$ and $(-3, \sqrt{7}/2)$, carefully indicating the correct direction and relationship with the level curves.

(b) Prove that for every point (x, y) the gradient vector $\nabla f(x, y)$ is orthogonal to the level curve of f through the point (x, y) by showing that the dot product of the tangent vector and the gradient vector equals zero.

5. (10 points) Find and classify (as local minimum, local maximum, or saddle point) every critical point of the function $f(x, y) = x^2y - 3x^2 - 6y^2 + 2$.

6. (5 points) Find the point on the ellipse $x^2+6y^2+3xy = 40$ with the largest x-coordinate.

7. (10 points) Find the volume of the region in the first octant that is inside the sphere $x^2 + y^2 + z^2 = 16$ and also inside the cylinder $x^2 + y^2 = 4x$.

8. (10 points) Let E be the solid lying inside the sphere $x^2 + y^2 + z^2 = 9$, outside the sphere $x^2 + y^2 + z^2 = 1$, above the xy-plane, below the cone $z = \sqrt{x^2 + y^2}$, and in the first octant. Compute $\displaystyle\iiint_E z\, dV$.

9. (10 points) Use an appropriate change of variables to evaluate the double integral of

$$f(x, y) = (x + y)e^{x^2 - y^2}$$

on the rectangle with vertices $(2, 0)$, $(1, 1)$, $(-1, -1)$, and $(0, -2)$.

10. (8 points) Let C be the quarter of the circle $x^2 + y^2 = 9$ going from $(0, 3)$ to $(-3, 0)$. Compute $\int_C x^2 y \, ds$.

11. (8 points) Let C be the boundary of the triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$, oriented counterclockwise. Let $\vec{F}(x,y) = \langle 3y^2, x^2y + \cos^8(y) \rangle$. Compute $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.

12. (9 points) Let C be the line segment from $(1,0,-1)$ to $(0,-2,2)$. Compute

$$\int_C (2xy + 6x^2)\,dx + (x^2 - y^3)\,dy + z^2\,dz.$$

Extra credit (5 points): Use a carefully written out $\varepsilon - \delta$ proof to prove that

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - 2xy^2}{x^2 + y^2} = 0.$$

Fall 2014 Math 211 Final Exam Sol'ns

1. Cross-product of normal vectors to planes gives a vector parallel to both planes: $\langle 0,0,1\rangle \times \langle 1,2,3\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \langle -2, 1, 0\rangle$

The parametric eq's for the line are $\boxed{x = 1-2t,\ y = 2+t,\ z = 3}$

2. Linear approximation:

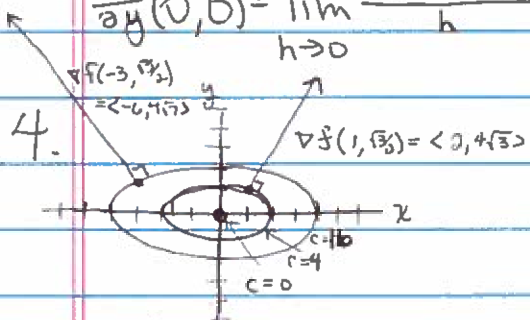
$$f(x,y,z) \cong f(2,1,2) + \frac{\partial f}{\partial x}(2,1,2)(x-2) + \frac{\partial f}{\partial y}(2,1,2)(y-1) + \frac{\partial f}{\partial z}(2,1,2)(z-2)$$
$$= 1 + \tfrac{1}{2}(x-2) + 1(y-1) - \tfrac{1}{2}(z-2)$$
$$f(2.04, 0.95, 2.02) \approx 1 + \tfrac{1}{2}(.04) + (-.05) - \tfrac{1}{2}(.02) = \boxed{0.96}$$

3. $\displaystyle \frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{2h^2/h^2 - 2}{h} = \lim_{h\to 0} \frac{0}{h} = 0$

$\displaystyle \frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{4h^2/2h^2 - 2}{h} = \lim_{h\to 0} \frac{0}{h} = 0$

$\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist since $f(x,y)\to 2$ along x-axis, $f(x,y)\to 3$ along $y=x$. Hence f is not continuous at $(0,0)$

4.



$\nabla f = \langle 2x, 8y\rangle$

Parameterization of level curve $x^2 + 4y^2 = c$ is

$\vec{r}(t) = \langle c\cdot\cos(t), \frac{c}{2}\cdot\sin(t)\rangle,\ 0 \le t \le 2\pi$

Tangent vector $\vec{r}\,'(t) = \langle -c\cdot\sin(t), \frac{c}{2}\cdot\cos(t)\rangle$

$\vec{r}\,'(t)\cdot\nabla f = \langle -c\cdot\sin(t), \frac{c}{2}\cdot\cos(t)\rangle \cdot \langle 2c\cdot\cos(t), 8\cdot\frac{c}{2}\cdot\sin(t)\rangle$

$= -2c^2\sin(t)\cos(t) + 2c^2\cos(t)\sin(t) = 0$ ✓

5. Critical points: solve $2xy - 6x = 0$ & $x^2 - 12y = 0$. Either $x=0$ or $y=3$, and $x^2 = 12y$.

Resulting points are $(0,0)$, $(\pm 6, 3)$.

At $(0,0)$, $D = 72$ and $f_{xx} = -6$, so $\boxed{(0,0) \text{ is a local max}}$

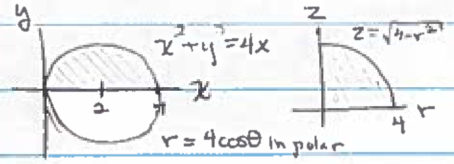
At $(\pm 6, 3)$, $D = -144$, so $\boxed{(\pm 6, 3) \text{ are saddle points}}$

6. Lagrange multipliers on $f(x,y) = x$, constraint $x^2 + 6y^2 + 3xy = 40$

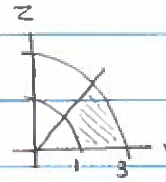
$\langle 1, 0\rangle = \langle 2x + 3y, 12y + 3x\rangle \cdot \lambda$

Solve $\begin{cases} 1 = \lambda(2x+3y) \\ 0 = \lambda(12y + 3x) \\ x^2 + 6y^2 + 3xy = 40 \end{cases}$ 2^{nd} eq gives $x = -4y$; substitute into constraint to find $y = \pm 2$. Candidate points are $(-8,2)$ and $(8,-2)$

$\boxed{\text{Largest } x\text{-coordinate on ellipse occurs at } (8,-2)}$

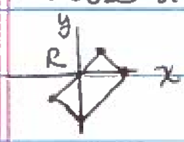
7. 

$$V=\int_0^{\pi/2}\int_0^{4\cos\theta}\sqrt{16-r^2}\,r\,dr\,d\theta$$
$$=\int_0^{\pi/2}\frac{1}{3}\left(64-64(1-\cos^2\theta)\sin\theta\right)d\theta$$
$$=\frac{64}{3}\left(\frac{\pi}{2}+\left(\cos\theta-\frac{1}{3}\cos^3\theta\right)\Big|_0^{\pi/2}\right)=\boxed{\frac{32\pi}{3}-\frac{128}{9}}$$

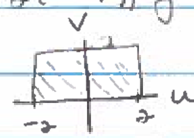
8. 

$$V=\int_0^{\pi/2}\int_{\pi/4}^{\pi/2}\int_1^3 \rho\cos\phi\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$=\frac{\pi}{2}\cdot\frac{1}{4}\rho^4\Big|_1^3\cdot\frac{1}{2}\sin^2\phi\Big|_{\pi/4}^{\pi/2}=\frac{\pi}{2}\cdot\frac{80}{4}\cdot\frac{1}{4}=\boxed{\frac{5\pi}{2}}$$

9. Choose $u=x+y$, $v=x-y$, so $x=\frac{1}{2}(u+v)$, $y=\frac{1}{2}(u-v)$ and $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\frac{1}{2}$



Boundaries
$y=x,\ y=x-2 \Rightarrow v=0,2$
$y=2-x,\ y=-2-x \Rightarrow u=2,-2$



$$\iint_R (x+y)e^{x^2-y^2}\,dA=\frac{1}{2}\int_{-2}^{2}\int_0^2 ue^{uv}\,dv\,du=\frac{1}{2}\int_{-2}^{2}(e^{2u}-1)\,du=\boxed{\frac{1}{4}(e^4-e^{-4})-2}$$

10. Parameterization of quarter circle: $\vec{r}(t)=\langle 3\cos(t), 3\sin(t)\rangle$, $\frac{\pi}{2}\le t\le\pi$
$$\int_C x^2y\,ds=\int_{\pi/2}^{\pi}27\cos^2(t)\sin(t)\cdot 3\,dt, \text{ using } \|\vec{r}\,'(t)\|=3$$
$$=-27\cos^3(t)\Big|_{\pi/2}^{\pi}=\boxed{27}$$

11. $$\int_C\vec{F}\cdot dr=\int_0^1\int_0^{2x}(2xy-6y)\,dy\,dx=\int_0^1(2x-6)\cdot\frac{1}{2}y^2\Big|_0^{2x}dx=x^4-4x^3\Big|_0^1=\boxed{-3}$$

using Green's Theorem

12. Potential function is $f(x,y,z)=x^2y+2x^3-\frac{1}{4}y^4+\frac{1}{3}z^3$
$$\int_C\vec{F}\cdot d\vec{r}=f(0,-2,2)-f(1,0,-1)$$
$$=-4+\frac{8}{3}-\left(2-\frac{1}{3}\right)=\boxed{-3}$$