Math 211 Multivariable Calculus Final Exam
Wednesday December 19, 2012

You have 3 hours for this exam. You may not use books, notes, calculators, cell phones or any other aids. Please turn off all electronic devices, including cell phones.

Explain your answers fully, showing all work in your blue book, and clearly label which problem you are solving. Answers involving sine, cosine or tangent of angles that are multiples of π, $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$, $\frac{\pi}{6}$ should be evaluated exactly. Follow directions carefully: If the question tells you to use a particular method or definition, then you must use it in order to get credit. The total number of points is 125.

1. (5 points) Prove that the four points $(0, 3, 2), (5, 6, 1), (1, 1, 2), (-5, 0, 3)$ are coplanar (lie on the same plane).

2. (12 points) Let
$$\mathbf{r}(t) = (3\sqrt{2}t)\mathbf{i} + e^{-3t}\mathbf{j} + e^{3t}\mathbf{k}$$
be the position of a particle at time t seconds.
   (a) Calculate the acceleration of this particle at time t.
   (b) Calculate the speed of this particle at time t. Please simplify. (Hint: the expression under the square root can be factored.)
   (c) Calculate the distance this particle travels between times $t = 0$ and $t = 1$ seconds.

3. (15 points) Let E be the region in $\mathbb{R}^3$ that lies above the plane $z = 0$, below the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = 9$.
   (a) Calculate the volume of E using spherical coordinates.
   (b) Calculate the volume of E using cylindrical coordinates.

4. (10 points) Let R be the region bounded by $x + y = 1$, $x + y = 4$, $x - y = -1$, and $x - y = 1$. Use an appropriate change of variables to calculate the integral
$$\iint_R (x + y)^2 e^{x-y} \, dA.$$

5. (10 points) Find the critical points of the function
$$f(x, y) = x^2 - x^2y + 2y^2$$
and classify each critical point as a local maximum, a local minimum or a saddle point.

6. (12 points) Let $f(x,y)$ be the function given by

$$f(x,y)=\begin{cases}\frac{3x^2}{\sqrt{x^2+y^2}} & \text{if } (x,y)\neq(0,0);\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

(a) Show that f is continuous at $(0,0)$.

(b) Calculate each of the partial derivatives of f at $(0,0)$, or show that it does not exist.

7. (12 points) Consider the function

$$f(x,y)=x^2+x\sin y.$$

(a) Find the directional derivative of f at the point $(1,0)$ in the direction of the vector $\langle 3,4\rangle$.

(b) What is the maximum value of any directional derivative of f at the point $(1,0)$?

(c) Use a linear approximation to estimate the value of $f(0.99,0.1)$.

8. (6 points) Either prove that the limit does not exist or calculate its value: $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^4+y^4}$.

9. (15 points) Find the absolute maximum and minimum of the function

$$f(x,y)=xy$$

subject to the constraint

$$(x-3)^2+y^2\leq 5.$$

(Explain how you know your values are the absolute maximum and minimum.)

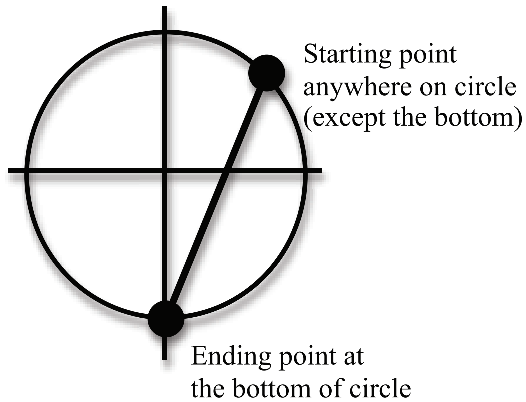
10. (6 points) Let $\mathbf{F}(x,y)=(2xy+y^3)\mathbf{i}+(x^2+3xy^2+2y)\mathbf{j}$. Show that the integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$ around any closed curve C in the xy-plane is zero.

11. (12 points) Verify Green's Theorem for the square D with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$, traversed once counterclockwise, for the functions $P(x,y)=-y$ and $Q(x,y)=x$ by evaluating both the appropriate line integral and double integral.

12. (10 points) Galileo's Paradox: The time for a bead to slide down a line segment starting at a point on a circle to the bottom of the circle is always the same, no matter where on the circle the line segment begins. The traversal time is given by the line integral

$$T = \int_C \frac{ds}{\sqrt{2g(y_0 - y)}},$$

where y_0 is the initial y-coordinate and g is the constant for gravitational acceleration. You may assume that the circle has radius 1. Find a parameterization for the line segment, and then compute the traversal time T to prove that it doesn't depend on the starting point on the circle.



Extra credit (5 points): Use an $\varepsilon - \delta$ proof to show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2(x^2+3) + y^2(y^2+3)}{x^2+y^2} = 3.$$