

# Fall 2017 Math 211 Exam 2 Review Sheet

The exam will be in class on Wednesday, November 1, and will cover Sections 14.1-14.8. You will not be allowed use of a calculator or any other device other than your pencil or pen and some scratch paper. Notes are also not allowed. In kindness to your fellow test-takers, please turn off all cell phones and anything that might beep or be a distraction.

Important topics:

- Level curves of  $z=f(x,y)$
- Limits and continuity for functions of 2 variables (no  $\epsilon$ - $\delta$  proofs on this exam)
- Computing partial derivatives from the definition (limit as  $h$  goes to 0 of the appropriate difference quotient)
- Partial derivatives
- Tangent plane to a surface
- Linear approximations to functions of 2 variables
- Chain rule
- Implicit differentiation of  $F(x,y,z)=0$
- Computing directional derivatives and interpretation as steepness of surface
- Gradient and its importance (points in direction of greatest increase), and its relation to level sets (gradient perpendicular to level curve or level surface)
- Second Partials Test to find and classify critical points of a function (local max/min and saddle points)
- Optimization of multivariable functions, including Lagrange multipliers

Chapter 14 Review Exercises: 5, 13, 15, 19, 25, 27, 29, 31, 33, 37, 44, 45, 51, 53, 55

Answer to 44(d):  $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \|\nabla f\| \|\mathbf{u}\| \cos \theta = \frac{1}{2} \|\nabla f\|$  and  $\mathbf{u}$  is a unit vector, so the directional derivative is half of its maximum value (length of the gradient) when the angle between  $\mathbf{u}$  and  $\|\nabla f\|$  is  $\theta = \cos^{-1}(1/2) = 60$  degrees or  $\pi/3$  radians.

Additional problems (answers on next page):

1.  $\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2 - y^2}{x^2 + 2y^2} = ?$

2.  $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(3x^2 + y^2)}{x^2 + 2y^2} = ?$

3.  $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^3 - y^3}{x^2 + y^2} = ?$

4. You are given only the following information about a differentiable function  $f$ :

$$f(3, 2) = 8, \quad f(3.01, 2) = 7.9, \quad f(3, 1.98) = 7.6.$$

- Approximate the equation of the tangent plane to the surface  $z=f(x,y)$  at  $(3, 2, 8)$ .
- Use part (a) to estimate the value of  $f(3.02, 1.96)$ .

5. Let  $f(x,y) = \begin{cases} \frac{5x^3 + xy^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$
- Evaluate  $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$  using polar coordinates and state whether this function is continuous at  $(0,0)$ .
  - Find  $\frac{\partial f}{\partial x}(0,0)$  and  $\frac{\partial f}{\partial y}(0,0)$ .
6. Find the directional derivative  $\mathbf{D}_v f(2,3)$  for  $f(x,y) = x^3y - 3x^2$  in the same direction as the vector  $\mathbf{v} = \langle 3, -4 \rangle$ . What can you say about the slope of the surface  $z=f(x,y)$  at  $(2,3,12)$  in the direction given by  $\mathbf{v}$ ?
7. Let  $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$
- Evaluate  $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$  using polar coordinates and state whether this function is continuous at  $(0,0)$ .
  - Find  $\frac{\partial f}{\partial x}(0,0)$  using the limit definition of partial derivative.
8. Find an equation for the tangent plane to the double-cone  $z^2 = x^2 + y^2$  at the point  $(a,b,c)$ , which could be any point on the cone (so your tangent plane equation will involve  $a, b$ , and  $c$  as well as  $x, y$ , and  $z$ ). Use this equation to show that the tangent plane passes through the origin, no matter which point  $(a,b,c)$  we chose on the cone.
9. The shape of a space probe entering the Earth's atmosphere is the ellipsoid  $4x^2 + y^2 + 4z^2 = 16$ . The probe's surface is heated by re-entry so that at a particular time the temperature is given by  $T(x,y,z) = 8x^2 + 4yz - 16z + 600$ . Find the hottest point(s) on the probe's surface.
10. Find the absolute maximum and minimum values of the function  $f(x,y,z) = x - y + z$  subject to the constraint  $x^2 + y^2 + z^2 = 1$ .
11. Find the local maxima, local minima, and saddle points of the function  $f(x,y) = x^2 + 6xy + 10y^2 - 4y + 4$ .

*Answers:* **1.** DNE (prove by finding two directions with different limits); **2.** DNE;  
**3.** 0 (can use polar coordinates); **4.** (a)  $z=8-10(x-3)+20(y-2)$  (b) 7.0; **5.** (a) Yes,  $f$  is continuous at  $(0,0)$  (b) Partial derivatives at  $(0,0)$  equal 5 and 0, respectively. **6.**  $\mathbf{D}_v f(2,3)=8$ , surface is sloping steeply up in this direction; **7.** (a) 0, is continuous, (b) 0. **8.** Note that  $c^2 = a^2 + b^2$  since  $(a,b,c)$  is on the cone. The gradient of  $F(x,y,z) = x^2 + y^2 - z^2$  is orthogonal to the surface  $F(x,y,z) = 0$ , so a normal vector at  $(a,b,c)$  is  $\langle 2a, 2b, -2c \rangle$ . The tangent plane is  $ax + by - cz = 0$ , and the point  $(0,0,0)$  satisfies this equation so lies on the tangent plane. **9.** Hottest points on surface are  $(4/3, -4/3, -4/3)$  and  $(-4/3, -4/3, -4/3)$ . **10.** Absolute max of  $\sqrt{3}$  at  $(1, -1, 1)/\sqrt{3}$  and absolute min of  $-\sqrt{3}$  at  $(-1, 1, -1)/\sqrt{3}$ . **11.**  $f(-6, 2) = 0$  is the local minimum (no other extrema).

## Practice Exam

- The equations  $z = f(x, y)$  and  $F(x, y, z) = f(x, y) - z = 0$  describe the same surface in 3-space, yet  $\nabla f$  and  $\nabla F$  are not the same vectors.
  - Write down  $\nabla f$  and  $\nabla F$  in terms of the partial derivatives of  $f$ .
  - If  $f(1,2) = 5$  and  $\nabla f(1,2) = 2\mathbf{i} - 3\mathbf{j}$ , find a vector normal to the surface  $z = f(x, y)$  at the point with  $x=1$  and  $y=2$ , and write down an equation for the tangent plane to the surface at that point.
- Find all points on the surface  $z = 3x^2 - 4y^2$  where the vector  $\langle 3, 2, 2 \rangle$  is perpendicular to the tangent plane.
- A flat circular plate is bounded by  $x^2 + y^2 = 4$ . The plate (including the boundary) is heated so that the temperature is  $T(x, y) = x^2 + 3y^2 + 6y + 12$  at each point  $(x, y)$  on the plate. Find the temperatures at the hottest and coldest points on the boundary  $x^2 + y^2 = 4$  of the plate.
- Suppose  $\nabla f = \langle -3, 0 \rangle$ .
  - In what direction(s) does the directional derivative  $\mathbf{D}_u f$  take its maximum value?
  - In what direction(s) does the directional derivative  $\mathbf{D}_u f$  take its minimum value?
  - In what direction(s) does the directional derivative  $\mathbf{D}_u f$  equal 0?
  - In what direction(s) does the directional derivative  $\mathbf{D}_u f$  equal half of its minimum value?

- Let  $f(x, y) = \begin{cases} \frac{x^3 - y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$ 
  - Evaluate  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  using polar coordinates and state whether this function is continuous at  $(0, 0)$ .
  - Find  $\frac{\partial f}{\partial x}(0, 0)$  and  $\frac{\partial f}{\partial y}(0, 0)$  using the limit definition of partial derivative.

**Brief answers:** 1.  $\nabla F = \langle f_x, f_y, -1 \rangle$  but  $\nabla f = \langle f_x, f_y \rangle$ ,  $2x - 3y - z + 9 = 0$ . 2.  $(-1/4, 1/8, 1/8)$ .

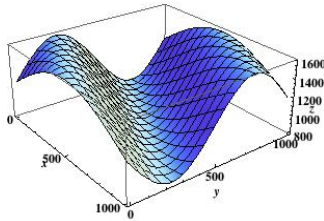
3. Hottest 36 at  $(0, 2)$ , coldest 11.5 at  $(\pm\sqrt{7}/2, -3/2)$ . 4a.  $\langle -1, 0 \rangle$  4b.  $\langle 1, 0 \rangle$  4c.  $\langle 0, 1 \rangle$  and  $\langle 0, -1 \rangle$  4d.  $\langle 1/2, \pm\sqrt{3}/2 \rangle$  5a. 0, continuous 5b.  $f_x(0, 0) = 1$  and  $f_y(0, 0) = -1$

## Practice Exam (no solutions available)

1. Suppose a region of vigorously rolling terrain can be modeled by

$$f(x,y) = 1200 + 400 \sin\left(\frac{\pi}{1000}x + \frac{\pi}{500}y\right),$$

where  $f(x,y)$  is the elevation in feet at the point  $(x,y)$ ,  $x$  is the distance east of  $(0,0)$ , and  $y$  is the distance north of  $(0,0)$ , both measured in feet.



Suppose you are located 600 feet east and 700 feet north of  $(0,0)$ .

- What is your elevation?
  - If you travel **west** from your location, what is the slope of the surface?
  - If you travel **north** from your location, what is the slope of the surface?
  - In what direction is the **steepest** slope? Please express as a unit vector.
  - In what direction(s) can you go to stay at the same elevation? Please express as unit vector(s).
2. You are given only the following information about a differentiable function  $f$ :
- $$f(2, -3) = 8, \quad f(2.01, -3) = 7.9, \quad f(2, -2.98) = 7.6.$$
- Approximate the equation of the tangent plane to the surface  $z=f(x,y)$  at  $(2, -3, 8)$ .
  - Use part (a) to estimate the value of  $f(1.98, -3.02)$ .

3. Let  $f(x,y) = \begin{cases} \frac{3x^3 + 5x^2y - 2y^3}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

- Evaluate  $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$  using polar coordinates and state whether this function is continuous at  $(0,0)$ .
  - Find  $\frac{\partial f}{\partial x}(0,0)$  and  $\frac{\partial f}{\partial y}(0,0)$ .
4. Find the local maxima, local minima, and saddle points of the function  $f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy$ .
5. Find an equation for the tangent plane to the cone  $z^2 = x^2 + y^2$  at the point  $(a,b,c)$ , which could be any point on the cone (so your tangent plane equation will involve  $a$ ,  $b$ , and  $c$  as well as  $x$ ,  $y$ , and  $z$ ). Use this equation to show that the tangent plane passes through the origin, no matter which point  $(a,b,c)$  we chose on the cone.