Spring 2019 Math 211 Exam 2 Review Sheet

The exam will be in class on Monday April 1, and will cover Sections 14.1-14.8. You will not be allowed use of a calculator or any other device other than your pencil or pen and some scratch paper. Notes are also not allowed. In kindness to your fellow test-takers, please turn off all cell phones and anything that might beep or be a distraction.

Important topics:

- Level curves of $z=f(x,y)$
- Limits and continuity for functions of 2 variables (no ε-δ proofs on this exam)
- Computing partial derivatives from the definition (limit as h goes to 0 of the appropriate difference quotient)
- Partial derivatives
- Tangent plane to a surface
- Linear approximations to functions of 2 variables
- Chain rule
- Computing directional derivatives and interpretation as steepness of surface
- Gradient and its importance (points in direction of greatest increase), and its relation to level sets (gradient perpendicular to level curve or level surface)
- Second Partials Test to find and classify critical points of a function (local max/min and saddle points)
- Optimization of multivariable functions, including Lagrange multipliers

Chapter 14 Review Exercises: 5, 13, 15, 19, 25, 27, 29, 31, 33, 37, 44, 45, 51, 53, 55, 59, 61, 63

Answer to 44(d): $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \|\nabla f\|\|\mathbf{u}\|\cos\theta = \frac{1}{2}\|\nabla f\|$ and **u** is a unit vector, so the directional derivative is half of its maximum value (length of the gradient) when the angle between **u** and $\|\nabla f\|$ is $\theta=\cos^{-1}(1/2)=60$ degrees or $\pi/3$ radians.

Additional problems (answers on next page):

1. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2 - y^2}{x^2 + 2y^2} = ?$
2. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(3x^2 + y^2)}{x^2 + 2y^2} = ?$
3. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^3 - y^3}{x^2 + y^2} = ?$

4. You are given only the following information about a differentiable function f:

   $f(3, 2) = 8, \quad f(3.01, 2) = 7.9, \quad f(3, 1.98) = 7.6.$

   a. Approximate the equation of the tangent plane to the surface $z=f(x,y)$ at (3, 2, 8).
   b. Use part (a) to estimate the value of $f(3.02, 1.96)$.

5. Let $f(x,y)=\begin{cases}\dfrac{5x^3+xy^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$
   a. Evaluate $\lim_{(x,y)\to(0,0)} f(x,y)$ using polar coordinates and state whether this function is continuous at (0,0).
   b. Find $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.
6. Find the directional derivative $\mathbf{D_u}f(2,3)$ for $f(x,y)=x^3y-3x^2$ in the same direction as the vector $\mathbf{v}=<3,-4>$. What can you say about the slope of the surface $z=f(x,y)$ at (2,3,12) in the direction given by **v**?
7. Let $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$
   a. Evaluate $\lim_{(x,y)\to(0,0)} f(x,y)$ using polar coordinates and state whether this function is continuous at (0,0).
   b. Find $\dfrac{\partial f}{\partial x}(0,0)$ using the limit definition of partial derivative.
8. Find an equation for the tangent plane to the double-cone $z^2=x^2+y^2$ at the point (a,b,c), which could be any point on the cone (so your tangent plane equation will involve a, b, and c as well as x, y, and z). Use this equation to show that the tangent plane passes through the origin, no matter which point (a,b,c) we chose on the cone.
9. The shape of a space probe entering the Earth's atmosphere is the ellipsoid $4x^2+y^2+4z^2=16$. The probe's surface is heated by re-entry so that at a particular time the temperature is given by $T(x,y,z)=8x^2+4yz-16z+600$. Find the hottest point(s) on the probe's surface.
10. Find the absolute maximum and minimum values of the function $f(x,y,z)=x-y+z$ subject to the constraint $x^2+y^2+z^2=1$.
11. Find the local maxima, local minima, and saddle points of the function $f(x,y)=x^2+6xy+10y^2-4y+4$.

Answers: **1**. DNE (prove by finding two directions with different limits); **2**. DNE; **3**. 0 (can use polar coordinates); **4**. (a) z=8-10(x-3)+20(y-2) (b) 7.0; **5**. (a) Yes, f is continuous at (0,0) (b) Partials at (0,0) equal 5 and 0, respectively. **6**. $\mathbf{D_u}f(2,3)=8$, surface is sloping steeply up is this direction; **7**. (a) 0, is continuous, (b) 0. **8**. Note that $c^2=a^2+b^2$ since (a,b,c) is on the cone. The gradient of $F(x,y,z)=x^2+y^2-z^2$ is orthogonal to the surface $F(x,y,z)=0$, so a normal vector at (a,b,c) is $<2a,2b,-2c>$. The tangent plane is $ax+by-cz=0$, and the point (0,0,0) satisfies this equation so lies on the tangent plane. **9.** Hottest points on surface are (4/3,-4/3,-4/3) and (-4/3,-4/3,-4/3). **10.** Absolute max of √3 at (1,-1,1)/√3 and absolute min of -√3 at (-1,1,-1)/√3. **11.** $f(-6,2)=0$ is the local minimum (no other extrema).

Practice Exam

1. The equations $z = f(x,y)$ and $F(x,y,z) = f(x,y) - z = 0$ describe the same surface in 3-space, yet ∇f and ∇F are not the same vectors.
    a. Write down ∇f and ∇F in terms of the partial derivatives of *f*.

    b. If $f(1,2) = 5$ and $\nabla f(1,2) = 2\mathbf{i} - 3\mathbf{j}$, find a vector normal to the surface $z = f(x,y)$ at the point with *x*=1 and *y*=2, and write down an equation for the tangent plane to the surface at that point.

2. Find all points on the surface $z = 3x^2 - 4y^2$ where the vector <3,2,2> is perpendicular to the tangent plane.

3. A flat circular plate is bounded by $x^2 + y^2 = 4$. The plate (including the boundary) is heated so that the temperature is $T(x,y) = x^2 + 3y^2 + 6y + 12$ at each point (*x*,*y*) on the plate. Find the temperatures at the hottest and coldest points on the boundary $x^2 + y^2 = 4$ of the plate.

4. Suppose $\nabla f = \langle -3, 0 \rangle$.
    a. In what direction(s) does the directional derivative $\mathbf{D_u}f$ take its maximum value?

    b. In what direction(s) does the directional derivative $\mathbf{D_u}f$ take its minimum value?

    c. In what direction(s) does the directional derivative $\mathbf{D_u}f$ equal 0?

    d. In what direction(s) does the directional derivative $\mathbf{D_u}f$ equal half of its minimum value?

5. Let $f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$
    a. Evaluate $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ using polar coordinates and state whether this function is continuous at (0,0).
    b. Find $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$ using the limit definition of partial derivative.

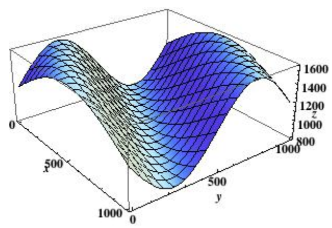
Brief answers: 1. $\nabla F = \langle f_x, f_y, -1\rangle$ but $\nabla f = \langle f_x, f_y \rangle$, $2x-3y-z+9=0$. **2.** (-1/4, 1/8, 1/8). **3.** Hottest 36 at (0,2), coldest 11.5 at $(\pm\sqrt{7}/2, -3/2)$. 4a. <-1,0> **4b**. <1,0> **4c**. <0,1> and <0,-1> **4d**. $\langle 1/2, \pm\sqrt{3}/2 \rangle$ **5a**. 0, continuous **5b**. $f_x(0,0)=1$ and $f_y(0,0)=-1$

Practice Exam (no solutions available)

1. Suppose a region of vigorously rolling terrain can be modeled by

$$f(x,y) = 1200 + 400\sin\left(\frac{\pi}{1000}x + \frac{\pi}{500}y\right),$$

where $f(x,y)$ is the elevation in feet at the point (x,y), x is the distance east of (0,0), and y is the distance north of (0,0), both measured in feet.



Suppose you are located 600 feet east and 700 feet north of (0,0).

   a. What is your elevation?
   b. If you travel **west** from your location, what is the slope of the surface?
   c. If you travel **north** from your location, what is the slope of the surface?
   d. In what direction is the **steepest** slope? Please express as a unit vector.
   e. In what direction(s) can you go to stay at the same elevation? Please express as unit vector(s).

2. You are given only the following information about a differentiable function f:

$$f(2, -3) = 8, \quad f(2.01, -3) = 7.9, \quad f(2, -2.98) = 7.6.$$

   a. Approximate the equation of the tangent plane to the surface $z=f(x,y)$ at (2, -3, 8).
   b. Use part (a) to estimate the value of $f(1.98, -3.02)$.

3. Let $f(x,y) = \begin{cases} \dfrac{3x^3 + 5x^2y - 2y^3}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

   a. Evaluate $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ using polar coordinates and state whether this function is continuous at (0,0).
   b. Find $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.

4. Find the local maxima, local minima, and saddle points of the function $f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy$.

5. Find an equation for the tangent plane to the cone $z^2 = x^2 + y^2$ at the point (a,b,c), which could be any point on the cone (so your tangent plane equation will involve a, b, and c as well as x, y, and z). Use this equation to show that the tangent plane passes through the origin, no matter which point (a,b,c) we chose on the cone.