Math 211 Practice Final Exam Problems

You are not allowed to use books, notes or calculators. You must explain your answers completely and clearly to get full credit.

1. Find an equation for the plane that contains the line of intersection of the planes $x+y+z=1$ and $3x-y-z=1$ and is perpendicular to the plane $x+2y+z=0$. *Answer: 3x-y-z=1.*

2. The trajectories of two particles are given by $\vec{r_1}(t)=\langle t,t^2,t^3\rangle$ and $\vec{r_2}(t)=\langle -1+3t,1+3t,-1+9t\rangle$.
   (a) Find all the intersection points of the two paths. *Answer: (-1,1,-1) and (2,4,8).*
   (b) Will the two particle collide? Explain. *Answer: No, there is no intersection with same value for t.*

3. Find the arc length of the curve $\vec{r}(t)=\langle \cos t,\sin t,t\rangle$, $0\le t\le 2\pi$. *Answer:* $s=2\sqrt{2}\pi$.

4. Consider the function
$$f(x,y)=\begin{cases}\dfrac{x^3}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$
   (a) Find $f_x(0,0)$ and $f_y(0,0)$. *Answer: 1 and 0.*
   (b) Recall that the directional derivative of $f(x,y)$ at (x_0,y_0) in the direction of the unit vector $\vec{u}=\langle a,b\rangle$ is defined as follows
$$D_{\vec{u}}f(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0+ah,y_0+bh)-f(x_0,y_0)}{h}.$$
   Use this to find $D_{\langle\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\rangle}f(0,0)$. *Answer:* $\frac{1}{2\sqrt{2}}$. *Note that this does not agree with the gradient dotted with the direction vector, which indicates that the function is not differentiable.*

5. (5 points) Find and classify (as local maximum, local minimum, or saddle point) all the critical points of $f(x,y)=x^4-4xy+2y^2$. *Answer: saddle at (0,0) and local minima at (1,1) and (-1,-1).*

6. Find the maximum and minimum values of $f(x,y)=x^2+y$ subject to the constraint $x^2+y^2=1$. *Answer: max value of* $\frac{5}{4}$ *at* $(\pm\frac{\sqrt{3}}{2},\frac{1}{2})$ *and min value of -1 at (0,-1).*

7. Calculate the volume of the region inside the sphere $x^2 + y^2 + z^2 = 2$, above the cone $z = \sqrt{x^2 + y^2}$ and in the first octant. *Answer:* $\frac{\pi}{3}(\sqrt{2} - 1)$.

8. (10 points) Evaluate

$$\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{0}^{9-x^2-y^2} x^2 + y^2 dz dy dx$$

by changing to cylindrical coordinates first. *Answer:* $\frac{\pi 3^5}{4}$.

9. Find the volume of the solid doughnut $z^2 + (\sqrt{x^2 + y^2} - b)^2 \leq a^2$ where $0 < a < b$. *Answer:* $2\pi^2 ba^2$. *Note: you will end up with the sum of two integrals; one can be done using u-substitution and the other by using a geometry formula to compute the corresponding area.*

10. Compute the area of the part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$. *Answer:* $\frac{2\pi}{3}(2\sqrt{2} - 1)$.

11. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y) = \langle y, x^2 \rangle$ and C is composed of the line segments from $(0, 0)$ to $(1, 0)$ and from $(1, 0)$ to $(1, 2)$. *Answer: 2.*

12. Use Green's theorem to compute $\int_C y dx + x dy$ where C is the boundary curve of the region bounded by $y = \sqrt{x}$, $y = 0$ and $x = 16$, in the counterclockwise direction. *Answer: 0.*

13. Evaluate the line integral $\int_C xy^4 ds$ where C is the upper half of the circle $x^2 + y^2 = 16$ oriented clockwise. *Answer: 0.*

14. Prove that

$$f(x, y) = \begin{cases} \dfrac{xy^4}{x^2 + y^8} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is not continuous at $(0, 0)$. *Answer: Find two directions yielding different limits as you approach (0,0).*

15. Evaluate the surface integral $\int_C xy^4 ds$ where C is the upper half of the circle $x^2 + y^2 = 16$ oriented clockwise. *Answer: 0.*

16. Let $\mathbf{F}(x, y, z) = \langle x^2 + y^2, 0, z^2 \rangle$ be the velocity field (in cm/sec) of a fluid. Compute the flow rate through the upper hemisphere S of the unit sphere centered at the origin.

Answer: $\pi/2$ *cm*^{3}*/sec (hint: integrate with respect to* θ *first to simplify the integral, then integrate with respect to* ϕ*).*

17. Use Stokes' Theorem to calculate the line integral of $\mathbf{F}(x, y, z) = \langle \sin(x^2), e^{y^2} + x^2, z^4 + 2x^2 \rangle$ around the triangle with vertices (3,0,0), (0,2,0), and (0,0,1), traversed counterclockwise when viewed from above. *Answer: 0.*

18. Use the Divergence Theorem to calculate the surface integral of the vector field $\mathbf{F}(x, y, z) = \langle y, yz, z^2 \rangle$ on the cylinder $x^2 + y^2 = 4$ and $0 \le z \le 5$ (surface includes top, bottom, and lateral sides, all with outward pointing normal vectors). *Answer:* 150π.

19. Use the Divergence Theorem to show that the flux of the vector field

$$\mathbf{F}(x, y, z) = \langle z^2 - xy^2, \cos(x + z), e^{-y} + zy^2 \rangle$$

through any smooth surface S equals 0.