Math 211 Curvature Practice

Formulas for curvature.

If $\mathbf{r}$ is a vector-valued function defining a smooth space curve C, and if $\mathbf{r}'(t)$ is not zero and if $\mathbf{r}''(t)$ exists, then the curvature κ of C satisfies

- $\kappa = \kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$
- $\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$.

1. Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b > 0$.
   a. Sketch this ellipse.
   b. Find a parameterization $\mathbf{r}(t)$ of this ellipse.
   c. Use of one of the two formulas above to find the curvature of this ellipse.
   d. At what points is the curvature of the ellipse greatest and at what points is it smallest? Why does this make sense?

Taken from *Active Calculus* by S. Schlicker, D. Austin, and M. Boelkins

2. Derivation of the second curvature formula:
    a. Explain why $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$, where s is arc length.
    b. Use part a and the definition of $\mathbf{T}(t)$ to derive $\mathbf{r}'(t) = \frac{ds}{dt}\mathbf{T}(t)$.
    c. Take a derivative of the equation in part b, using product rule.
    d. Combine parts b and c to derive $\mathbf{r}'(t)\times\mathbf{r}''(t) = \left(\frac{ds}{dt}\right)^2 \mathbf{T}(t)\times\mathbf{T}'(t)$.
    e. Show that $\|\mathbf{T}(t)\times\mathbf{T}'(t)\| = \|\mathbf{T}'(t)\|$.
    f. Use the first curvature formula and the parts above to derive the second formula.

Taken from *Active Calculus* by S. Schlicker, D. Austin, and M. Boelkins