Spring 2013 Math 260 Exam 2 Review Sheet

The exam will be in class on Monday, April 22, and will cover Chapters 4-6.

You will not be allowed use of a calculator or any other device other than your pencil or pen and some scratch paper. Notes are also not allowed. In kindness to your fellow test-takers, please turn off all cell phones and anything else that might beep or be a distraction.

Important topics:

- Flow on a circle (finding fixed points and determining stability)
- Determining types of bifurcations occurring for flow on a circle
- Calculating time of bottlenecks near a saddle-node bifurcation
- Classify fixed points as Liapunov stable, attracting, or asymptotically stable.
- Classification of fixed points of a linear system using eigenvalues
- Classification of fixed points of a nonlinear system using eigenvalues of the linearized system, including systems involving parameters
- Sketching phase portraits of linear and nonlinear systems (e.g., sketch nullclines, fixed points, and use eigenvectors to determine directions of the stable and unstable manifolds of saddle points)
- Rescale a system of differential equations to be dimensionless
- Conservative systems and deriving conserved quantities
- Homoclinic orbits
- Reversible systems
- Theorems for conservative and reversible systems indicating when purely imaginary eigenvalues for a fixed point of a nonlinear system do indicate a center

If you have questions about certain topics, feel free to stop by my office.

Practice Exam (longer than the actual):

1. Find and classify all the fixed points of $\dot{\theta} = -\sin^3(3\theta)$, then sketch the phase portrait on the circle.

2. Consider the flow $\dot{\theta} = \mu - 2\cos 2\theta$ on the circle, where μ is a parameter that can be positive, negative, or zero. Draw the different possible types of phase portraits that can occur for this flow on the circle, and classify the bifurcations that occur (giving the critical values of μ and θ for each bifurcation).

3. Classify the fixed point of the linear system $\dot{x} = -3x + 2y$, $\dot{y} = x - 2y$.

4. Consider the circuit equation $L\ddot{I} + R\dot{I} + I/C = 0$, where L and C are positive parameters and $R \geq 0$.
   a. Rewrite the equation as a system in the plane.
   b. Show that the origin is asymptotically stable if $R>0$ but only Liapunov stable if $R=0$.
   c. Classify the fixed point at the origin, depending on whether R^2C-4L is positive, negative, or zero, and sketch the phase portrait in each case.

5. Find and classify the fixed points of the nonlinear system $\dot{x} = y + x - x^3$, $\dot{y} = -y$. Sketch the phase portrait, where the stable and unstable manifolds are drawn in the proper directions near each saddle point.

6. Find a conserved quantity for the Duffing equation $\ddot{x} + x + \varepsilon x^3 = 0$. Use it to show that the origin is a nonlinear center.

7. Show that the system $\ddot{x} + x\dot{x}^2 + x = 0$ is reversible, and plot the phase portrait.

8. A type of biochemical switch describing the expression of a gene can be modeled via $\dot{g} = k_1 s_0 - k_2 g + \dfrac{k_3 g^2}{k_4 + g^2}$, where all parameters are positive and $g(t)$ is the concentration of the gene product. Show that this differential equation can be put in the much more manageable dimensionless form $\dfrac{dx}{d\tau} = s - rx + \dfrac{x^2}{1+x^2}$. Clearly define x, τ, s, and r, and demonstrate how to derive the new equation.

Partial solutions:

1. Six fixed points, alternating in stability with θ=0 being stable.
2. Four saddle-node bifurcations occur: μ_c=2, θ*=0 or π; : μ_c=-2, θ*= π /2 or 3π/2.
3. The origin is a stable node (sink).
4. If R^2C-4L is positive, then the origin is a stable node (with unequal eigenvalues); if R^2C-4L is negative, then the origin is a stable spiral; if R^2C-4L is zero, then the origin is a stable node with repeated eigenvalue with a single eigenvector (so degenerate).
5. (0,0) is a saddle point; (1,0) are stable nodes. A reasonable guess is sufficient for the phase portrait, with the stable and unstable manifolds indicated near (0,0) using the eigenvectors. It may help to fill in a bit of the vector field to get a decent idea of the general phase portrait—check your answer using Mathematica.
6. $E(x,y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{\varepsilon}{4}x^4$, taking its minimum value at the origin (for any value of ε; check using calculus). The theorem on conservative systems tells us that trajectories near the origin must then be closed orbits.
7. The system is invariant under the transformation $t \to -t$, $y \to -y$. The only fixed point is the origin. The linearization has purely imaginary eigenvalues, so the origin is a linear center. Since the system is reversible, the theorem from section 6.6 implies that the origin is in fact a nonlinear center. Hence all trajectories near the origin are closed orbits.
8. Let $x = g/\sqrt{k_4}$, $\tau = k_3 t/\sqrt{k_4}$, $s = k_1 s_0 / k_3$, $r = k_2\sqrt{k_4}/k_3$.

Please let me know if you find any errors in these solutions.