Math 260 Exam 3: Take-Home

Due Wednesday, May 15, at 4pm.

You may not discuss the exam with anyone except me, but you may use the textbook, your notes from class, and *Mathematica* (including any files you created in Math 260 labs or that are posted on the course webpage). You may not use any other materials. In particular, the internet is off-limits (except files on the course webpage).

Please answer each question clearly and fully. Submit a *Mathematica* file with any work you do in *Mathematica* (required for Exercises 3 and 4, optional for the others).

1. (20pt) Consider the system $\dot{x} = -x^3 + xy^2$, $\dot{y} = -2x^2y - y^3$. Find the fixed point, then construct a Liapunov function (imitating the forms we have used previously) to show that this fixed point is globally asymptotically stable.

2. (30pt) Find and classify the bifurcation for the following system:

$$\begin{aligned} \dot{x} &= y - x \\ \dot{y} &= -by + \frac{x}{1+x} \end{aligned}$$

   You may assume that the parameter b is positive. Specify the value of b at which the bifurcation occurs.

3. (35pt) Consider the following predator-prey system (with prey $x \geq 0$, predator $y \geq 0$, and parameter $a > 0$):

$$\begin{aligned} \dot{x} &= x(x - x^2 - y) \\ \dot{y} &= y(x - a) \end{aligned}$$

   (a) Sketch the null clines in the 1st quadrant for the 2 cases $a > 1$ and $a < 1$ (indicating direction of flow).

   (b) Find the fixed points. Classify the origin using your sketch of the null clines, and classify the other points using eigenvalues of the linearization (for values of a such that they remain in the 1st quadrant).

   (c) Sketch the phase portrait for $a > 1$ and argue that the predators will go extinct.

   (d) Show that, for a in some interval around $1/2$, the eigenvalues of one of the fixed points form a complex conjugate pair, with the real part changing signs at $a_c = 1/2$.

   (e) The preceding part suggests that a Hopf bifurcation occurs at $a_c = 1/2$. Plot trajectories showing the existence of an attracting limit cycle that shrinks downs to the fixed point and then disappears as a approaches $1/2$, leaving an attracting fixed point. That is, demonstrate (using graphs in Mathematica) that a supercritical Hopf bifurcation occurs.

(f) Estimate the frequency of limit cycle oscillations for a near the bifurcation value (using the eigenvalues).

(g) Sketch all of the qualitatively different phase portraits for $0 < a < 1$.

4. (15pt) Imitating the numerical calculations done in lecture (the *Mathematica* file is available on the course webpage), estimate the largest Liapunov exponent for the Lorenz system with $r = 28$, $\sigma = 10$, and $b = 8/3$.