Spring 2013 Math 272 Exam 2 Review Sheet

The exam will be in class on Wednesday, April 3, and will cover Sections 3.1-4.4.

You will not be allowed use of a calculator or any other device other than your pencil or pen and some scratch paper. Notes are also not allowed. In kindness to your fellow test-takers, please turn off all cell phones and anything else that might beep or be a distraction.

Important topics:

- Determinants
- Eigenvalues and eigenvectors
- Application to matrix powers
- Vector spaces and subspaces (of vectors, matrices, and polynomials)
- Span
- Linear independence
- Basis

If you have questions on certain topics or want more practice on exam-like problems, please feel free to stop by my office.

Chapter 3 review (p227): 4, 7a, 8a-f, 10, 12, 18, 19, 20

Chapter 4 review (p316): 3, 5, 6, 8, 9, 13, 15, 16, 20, 21, 22, 27

<u>Extra review problems:</u>

1. Prove that $|\mathbf{A}^{-1}|=1/|\mathbf{A}|$ for all invertible matrices **A**.

2. Prove that if λ is an eigenvalue of a matrix **A** with corresponding eigenvector **v**, then λ^m is an eigenvalue of $\mathbf{A}^m$ with corresponding eigenvector **v**.

3. Prove that if **A** is nilpotent ($\mathbf{A}^m=0$ for some positive integer m), then 0 is an eigenvalue of **A**. In fact, show that 0 is the *only* eigenvalue of **A**.

4. Let **A** be a 2x2 matrix with eigensystem $\left\{\left\{\lambda_1 = -\frac{1}{2}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}, \left\{\lambda_2 = 1, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}\right\}$

   and let $\mathbf{x}_0 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. Find the limit of $\mathbf{x}_k=\mathbf{A}^k\mathbf{x}_0$ as $k\to\infty$.

5. Suppose a 2x2 matrix **A** has eigensystem $\left\{\left\{\lambda_1 = 1,\ \mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}, \left\{\lambda_2 = \frac{2}{3},\ \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}\right\}$.

   Determine $\lim_{k\to\infty} \mathbf{A}^k \mathbf{x}_0$ if $\mathbf{x}_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$.

6. Find a basis for the vector space $\left\{p(x) = a_0 + a_1 x + a_2 x^2 : p(1) = 0\right\}$. What is the dimension of this space?

7. Find a basis for the space $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + b - c = 0 \text{ and } b + 2d = 0\right\}$. What is the dimension of this space?

8. Prove that $B = \left\{1 + x, x + x^2, 1 - x\right\}$ is a basis for $\mathbf{P}_2$. Express $2+x+x^2$ as a linear combination of the vectors in this basis.

9. Determine if the set S is a subspace of the vector space V. If it is, then find a basis for S. If it is not, then explain why not.

   (a) $S = \left\{\begin{pmatrix} a & b \\ a & b \end{pmatrix} : a,b \text{ are real}\right\}$ for V = set of all 2x2 matrices with real entries

   (b) $S = \left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + z = 1\right\}$ for $V=\mathbf{R}^3$

<u>Partial solutions</u>

1. Let **A** be an invertible matrix, so $\mathbf{A}\mathbf{A}^{-1}=\mathbf{I}$. Then $1=|\mathbf{I}|=|\mathbf{A}\mathbf{A}^{-1}|=|\mathbf{A}||\mathbf{A}^{-1}|$, so $|\mathbf{A}^{-1}|=1/|\mathbf{A}|$. Note that we can divide by $|\mathbf{A}|$ since invertible matrices are nonsingular ($|\mathbf{A}|\neq 0$).
2. Suppose $\mathbf{A}\mathbf{v}=\lambda\mathbf{v}$. We will prove that $\mathbf{A}^m\mathbf{v}=\lambda^m\mathbf{v}$ for any positive integer m by induction. The base case is $m=1$, and we already know $\mathbf{A}\mathbf{v}=\lambda\mathbf{v}$. Suppose $\mathbf{A}^m\mathbf{v}=\lambda^m\mathbf{v}$ for some positive integer m. Multiply both sides of this equation by A to obtain $\mathbf{A}^{m+1}\mathbf{v}=\lambda^m\mathbf{A}\mathbf{v}=\lambda^m\lambda\mathbf{v}=\lambda^{m+1}\mathbf{v}$. Hence the statement is also true for $m+1$. Therefore, by induction the statement is true for all positive integers m.

3. Suppose $\mathbf{A}^m=\mathbf{0}$ for some positive integer m. Then $|\mathbf{A}|^m=|\mathbf{A}^m|=0$, so $|\mathbf{A}-0\mathbf{I}|=|\mathbf{A}|=0$. This says that $\lambda=0$ satisfies the characteristic equation of $\mathbf{A}$, so is an eigenvalue of $\mathbf{A}$. Now suppose that λ is any eigenvalue of $\mathbf{A}$. For some nonzero vector v, $\mathbf{Av}=\lambda\mathbf{v}$. From #2, $\mathbf{A}^m\mathbf{v}=\lambda^m\mathbf{v}$. But $\mathbf{A}^m=\mathbf{0}$, so $\lambda^m\mathbf{v}=\mathbf{A}^m\mathbf{v}=\mathbf{0v}=\mathbf{0}$. We know $\mathbf{v}$ is nonzero, so it must be that $\lambda=0$. Hence $\lambda=0$ is the only eigenvalue of $\mathbf{A}$.
4. Write $\mathbf{x}_0$ as a linear combination of eigenvectors then multiply by $\mathbf{A}^k$ and simplify using eigenvalues. The limit is $\begin{bmatrix}4\\2\end{bmatrix}$.
5. The limit is $\begin{bmatrix}3/4\\1/4\end{bmatrix}$.
6. A basis is $B=\{x-1, x^2-1\}$, so dimension is 2.
7. A basis is $B=\left\{\begin{bmatrix}1&0\\1&0\end{bmatrix},\begin{bmatrix}2&-2\\0&1\end{bmatrix}\right\}$, so dimension is 2.
8. To prove the set is a basis, we first note that it contains 3 vectors and $\mathbf{P}_2$ is a 3 dimensional space, so we only need to prove it spans $\mathbf{P}_2$ or that it is linearly independent. To do the latter, show that the only linear combination of the 3 polynomials that equals zero has all 3 coefficients equal to zero. $2+x+x^2$ equals the sum of the 3 basis polynomials.
9. (a) Yes, this is a subspace; (b) Not a subspace since doesn't contain zero vector.

Please email me if you find any errors in the solutions. Thank you!

<u>Practice exam:</u>

1. Compute $\begin{vmatrix}1&0&0\\p&1&s\\q&r&2\end{vmatrix}$. For what values of p, q, r, and s is the following set a basis for $\mathbf{R}^3$: $\left\{\begin{bmatrix}1\\p\\q\end{bmatrix},\begin{bmatrix}0\\1\\r\end{bmatrix},\begin{bmatrix}0\\s\\2\end{bmatrix}\right\}$

2. Prove that $B=\{1-x,1+x,x-x^2\}$ is a basis for $\mathbf{P}_2$. Express $3x^2+x$ as a linear combination of the vectors in this basis.

3. Find a basis for the space $\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}: a+3b=0 \text{ and } c+3d=0\right\}$. What is the dimension of this space?

4. Suppose V is a vector space and $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is a basis for V. Prove that $\{\mathbf{u},\mathbf{u}+\mathbf{v},\mathbf{u}+\mathbf{v}+\mathbf{w}\}$ is also a basis for V.

5. Prove that $\lambda=0$ is an eigenvalue of a matrix $\mathbf{A}$ if and only if $\mathbf{A}$ is singular.