Math 272 Practice Problems Involving Linear Transformations

1. Suppose that $T : V \to W$ is a linear transformation. Prove that T is one-to-one if and only if the only solution to $T(\mathbf{v}) = \mathbf{0}$ is $\mathbf{v} = \mathbf{0}$.

2. For each of the following transformations, determine the kernel and the range and whether the transformation is one-to-one and/or onto.

   (a) $T : \mathbf{R}^2 \to \mathbf{R}^2$, $T(x, y) = (2x - 3y, 5x + y)$.

   (b) $T : \mathbf{R}^2 \to \mathbf{R}^2$, $T(x, y) = (8x + 4y, 2x + y)$.

   (c) $T : \mathbf{R}^3 \to \mathbf{R}^2$, $T(x, y, z) = (x - y, y - z)$.

   (d) $T : \mathbf{R}^2 \to \mathbf{R}^3$, $T(x, y) = (2x - 3y, 5x + y, y)$.

3. Suppose that a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ is defined by $T\mathbf{v} = \mathbf{A}\mathbf{v}$, where $\mathbf{A}$ is an $n \times n$ matrix with $\det\mathbf{A}{=}0$. Can T be one-to-one? Can T be onto? Explain.

4. If $\mathbf{A}$ is an $m \times n$ matrix and the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ has at least one solution for every vector $\mathbf{b}$ in $\mathbf{R}^m$, what is the range of $T_{\mathbf{A}} : \mathbf{R}^n \to \mathbf{R}^m$?

5. Let T be a transformation from the set of polynomials of degree 2 or less to $\mathbf{R}^2$ defined by $T(p) = (p(1), p(-1))$. What are the range and kernel of T? Is T one-to-one? Is it onto?

6. If $T : V \to W$ is a linear transformation that is both one-to-one and onto, then for each vector $\mathbf{w}$ in W there is a unique vector $\mathbf{v}$ in V such that $T(\mathbf{v}) = \mathbf{w}$. Prove that the inverse transformation $T^{-1} : W \to V$ defined by $T^{-1}(\mathbf{w}) = \mathbf{v}$ is linear.

7. Let V be a vector space and let $\mathbf{v}_1, \mathbf{v}_2 \in V$. This gives subspaces $\text{Span}\{\mathbf{v}_1\}$ and $\text{Span}\{\mathbf{v}_2\}$ of V. Prove that $\text{Span}\{\mathbf{v}_1\} + \text{Span}\{\mathbf{v}_2\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where

$$\text{Span}\{\mathbf{v}_1\} + \text{Span}\{\mathbf{v}_2\} = \{\mathbf{u} + \mathbf{v} |\ \mathbf{u} \in \text{Span}\{\mathbf{v}_1\}, \mathbf{v} \in \text{Span}\{\mathbf{v}_2\}\}.$$

8. Let V be a vector space and assume that $\mathbf{v}_1, \mathbf{v}_2 \in V$ are linearly independent. Given $\mathbf{v}_3 \in V$, prove that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent if and only if $\mathbf{v}_3 \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Prove this directly from the definitions.

9. Let V be a vector space and assume that $\mathbf{v}, \mathbf{w} \in V$. Prove that $\text{Span}\{\mathbf{v}, \mathbf{w}\} = \text{Span}\{\mathbf{v}+\mathbf{w}, \mathbf{v} - \mathbf{w}\}$.

10. Prove that the kernel of a linear transformation $T : V \to W$ is a subspace of V. You should prove this directly from the appropriate definitions.

11. Prove that the range of a linear transformation $T : V \to W$ is a subspace of W. You should prove this directly from the appropriate definitions.

12. Let $T : V \to W$ be linear and one-to-one. Suppose $\mathbf{v}_1, \cdots, \mathbf{v}_n \in V$ are linearly independent. Prove that $T(\mathbf{v}_1), \cdots, T(\mathbf{v}_n)$ are linearly independent. You should prove this directly from the appropriate definitions.

13. Let $T : V \to W$ be linear and let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be a basis of V. Also assume that $T(\mathbf{v}_1), \cdots, T(\mathbf{v}_n)$ are linearly independent. Prove that T is one-to-one. You should prove this directly from the appropriate definitions.

14. Let $T : V \to W$ and $S : W \to Z$ be linear maps between vector spaces. Prove that the composition $S \circ T : V \to Z$ is linear. You should prove this directly from the appropriate definitions.

15. Let $T : V \to W$ be linear and onto. Also assume that $\mathbf{v}_1, \cdots, \mathbf{v}_n$ span V. Prove that $T(\mathbf{v}_1), \cdots, T(\mathbf{v}_n)$ span W. You should prove this directly from the appropriate definitions.

16. Let $T : V \to W$ be linear. Also assume that $\mathbf{v}_1, \cdots, \mathbf{v}_n \in V$ have the property that $T(\mathbf{v}_1), \cdots, T(\mathbf{v}_n)$ span W. Prove that T is onto. You should prove this directly from the appropriate definitions.

Selected Solutions

1. First suppose T is one-to-one, so $T(u) = T(v)$ implies $u = v$. For any linear transformation, $T(0) = 0$. If $T(u) = 0$, then $T(v) = T(0)$, so we must have $v = 0$.

   Now suppose the only solution to $T(v) = 0$ is $v = 0$. If $T(u) = T(v)$, then $T(u-v) = 0$ (using linearity of T) and so $u - v = 0$ (using the assumption), that is, $u = v$. This implies that T is one-to-one.

2. (a) range(T)=$\mathbb{R}^2$, ker(T)=$\{\mathbf{0}\}$. T is both one-to-one and onto (so invertible).
   (b) range(T)=Span$\{[4, 1]\}$, ker(T)=Span$\{[-1, 2]\}$. T is neither one-to-one nor onto.
   (c) range(T)=$\mathbb{R}^2$, ker(T)=Span$\{[1, 1, 1]\}$. T is onto but not one-to-one.
   (d) range(T)=Span$\{[2, 5, 0], [-3, 1, 1]\}$, ker($T$)=$\{\mathbf{0}\}$. T is one-to-one but not onto.

3. $Tv = Av$ where $|A| = 0$ implies that T is singular so neither one-to-one nor onto. In particular, the kernel of T and the null space of A contain more than the zero vector, so T is not one-to-one. Also, the columns of A must be linearly dependent since $|A| = 0$, so the n columns do not span $\mathbb{R}^n$, implying that range(T)=col(A) is not all of $\mathbb{R}^n$ and T cannot be onto.

4. The range of T_A must be $\mathbb{R}^m$ since for every $b \in \mathbb{R}^m$ we can find $x \in \mathbb{R}^n$ such that $Ax = b$.

5. $T(a_2x^2 + a_1x + a_0) = (a_2 + a_1 + a_0, a_2 - a_1 + a_0)$. The range of T is all of $\mathbb{R}^2$, and the kernel of T is Span$\{x^2 - 1\}$. T is not one-to-one, but it is onto.

6. Take any $w_1, w_2 \in W$ and scalar c. Let $v_1 = T^{-1}(w_1)$ and $v_2 = T^{-1}(w_2)$. Using the linearity of T and that $T^{-1}(T(v)) = v$ for all $v \in V$,

$$\begin{aligned} T^{-1}(cw_1 + w_2) &= T^{-1}(cT(v_1) + T(v_2)) \\ &= T^{-1}(T(cv_1 + v_2)) \\ &= cv_1 + v_2 \\ &= c\,T^{-1}(T(v_1)) + T^{-1}(T(v_2)) \\ &= c\,T^{-1}(w_1) + T^{-1}(w_2). \end{aligned}$$

   Therefore T^{-1} is linear.

7. First take any $u \in \text{Span}\{\mathbf{v_1}\} + \text{Span}\{\mathbf{v_2}\}$, so we can write $u = u_1 + u_2$ for some vectors $u_1 \in \text{Span}\{\mathbf{v_1}\}$ and $u_2 \in \text{Span}\{\mathbf{v_2}\}$. We can write $u_1 = c_1v_1$ and $u_2 = c_2v_2$ for some scalars c_1 and c_2, so $u = c_1v_1 + c_1v_2 \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Therefore $\text{Span}\{\mathbf{v_1}\} + \text{Span}\{\mathbf{v_2}\} \subseteq \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

   Now take any $u \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, so $u = c_1v_1 + c_1v_2$ for some scalars c_1 and c_2. Let $u_1 = c_1v_1 \in \text{Span}\{\mathbf{v_1}\}$ and $u_2 = c_2v_2 \in \text{Span}\{\mathbf{v_2}\}$. Then $u = u_1 + u_2 \in \text{Span}\{\mathbf{v_1}\} + \text{Span}\{\mathbf{v_2}\}$. Therefore $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq \text{Span}\{\mathbf{v_1}\} + \text{Span}\{\mathbf{v_2}\}$.

   We conclude that $\text{Span}\{\mathbf{v_1}\} + \text{Span}\{\mathbf{v_2}\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.