Math 294 Exam 1 (Take-home)

Please answer each problem as clearly and completely as you can. Do not discuss these problems with other students, or anyone else but me. You may use your textbook, lecture notes, class materials (including those posted on the website), and homework, but do not use other books, the internet, or any materials other than those directly associated with the course. Please do feel free to ask me questions, either via email or coming by my office. Show all work to demonstrate that you understand your answer. You may use computational aids like Mathematica and calculators (indicate how you did so in your write-up).

This exam is due **4pm Fri March 10**. Late submissions will be penalized 10 points per day. There are 5 problems with a total of 100 points.

Problem 1 (20pt) Which of the following functions is convex on $\mathbb{R}^n$? For each function, either prove that it is convex or explain why it is not.

(a) $f(x) = \|x\|_\infty$

(b) $f(x) = \sqrt{\|x\|_2}$

(c) $f(x) = \max\left\{ \sum_{k=1}^n x_k,\ \sum_{k=1}^n (-1)^k x_k \right\}$

(d) $f(x) = \min\left\{ \sum_{k=1}^n x_k,\ \sum_{k=1}^n (-1)^k x_k \right\}$

Problem 2 (20pt) *Monotropic property of convex functions*: Suppose $f : \mathbb{R} \to \mathbb{R}$ is a convex function. Use the definition of convexity to prove that if $x_1, x_2, x_3 \in \mathbb{R}$ satisfy $x_1 < x_2 < x_3$, then

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

Hint: solve $x_2 = \lambda x_1 + (1 - \lambda)x_3$ for λ and apply definition of convex function.

Problem 3 (20pt) A nonempty set C is a *cone* if $\lambda x \in C$ for every $x \in C$ and every $\lambda \geq 0$. Prove that a cone C is convex if and only if $C + C \subseteq C$ (that is, adding any two elements of the cone C produces another element also in C.)

Problem 4 (20pt) Find the subdifferential $\partial f(x)$ of the function $f(x) = |3x_1 - x_2 + 1|$ for $x = (x_1, x_2) \in \mathbb{R}^2$. Verify directly that for x satisfying $f(x) = 0$ we have

$$f(y) - f(x) - g_x^T(y - x) \geq 0$$

for all $g_x \in \partial f(x)$ and $y \in \mathbb{R}^2$.

Problem 5 (20pt) Sketch the feasible set and then solve the following LP:

$$p^* = \min_{x \in \mathbb{R}^2}(x_2 - 2x_1) \quad \text{s.t.} \quad 2x_1 + x_2 \leq 440,\ 4x_1 + x_2 \leq 680,\ x_1 \geq 0, \text{ and } x_2 \geq 0.$$