Math 294 Exam 2 (Take-home)

Please answer each problem as clearly and completely as you can. Do not discuss these problems with other students, or anyone else but me. You may use your textbook, lecture notes, class materials (including those posted on the website), and homework, but do not use other books, the internet, or any materials other than those directly associated with the course. Please do feel free to ask me questions, either via email or coming by my office. Show all work to demonstrate that you understand your answer. You may use computational aids like Mathematica and calculators (indicate how you did so in your write-up).

This exam is due **4pm Monday April 17**. Late submissions will be penalized 10 points per day. There are 5 problems with a total of 100 points.

Problem 1 (20pt) Consider the optimization problem

$$p^* = \min_{x \in \mathbb{R}^2} (-3x_1 - 2x_2)$$

$$\begin{aligned} \text{s.t.} \quad 5x_1 + 2x_2 &\le 90, \\ 4x_1 + 5x_2 &\le 140, \\ x_1, x_2 &\ge 0. \end{aligned}$$

(a) Sketch the feasible set and directly solve the primal problem, stating p^* and x^*.

(b) Find and then solve the dual problem, sketching the constrained region in the $\lambda_1\lambda_2$-plane and stating d^* and λ^*.

(c) Describe the connection between whether each Lagrange multiplier equals zero or not and whether each corresponding constraint is active or not.

Problem 2 (20pt) Consider the optimization problem

$$p^* = \min_{x \in \mathbb{R}^2} (x_1^2 + x_2^2) \quad \text{s.t.} \quad x_1 + 2x_2 \ge 2.$$

(a) Directly solve the primal problem, stating p^* and x^*.

(b) Find and then solve the dual problem, stating x_λ, d^*, and λ^*.

(c) Use x_λ and λ^* to recover the primal problem solution.

Problem 3 (20pt) Consider the optimization problem

$$p^* = \min_{x \in \mathbb{R}} (-2x) \quad \text{s.t.} \quad f_1(x) \le 0,$$

where

$$f_1(x) = \begin{cases} x - 3 & \text{if } x \ge 2 \\ -x/2 & \text{if } x < 2 \end{cases}$$

State and solve the primal and dual problems (determine p^*, x^*, d^*, and λ^*).

Problem 4 (20pt) Consider the optimization problem

$$p^* = \min_{x \in \mathbb{R}^4} \left(\frac{1}{2} x^T H x + c^T x \right) \quad \text{s.t.} \quad Ax = b,$$

where

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad c = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & -2 & 2 & 4 \\ 2 & 0 & 4 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

(a) Find the general solution $x = \tilde{x} + Nz$ to the linear system $Ax = b$.

(b) Substituting the general solution to $Ax = b$ into the quadratic objective function and state the resulting unconstrained QP in z. Verify that the new H matrix in the quadratic form for z is positive definite (so yields a convex function).

(c) Solve the new QP and use it to obtain the solution to the original problem. State the optimal z^*, x^*, and p^*.

Problem 5 (20pt) Suppose we want to fit a plane $z = ax + by + c$ to the set of points $\{(-1, 1, -1), (0, 1, 0), (1, 3, 1), (2, 3, 4)\}$.

(a) Solve the least squares problem that minimizes the 2-norm of the residual by computing the solution to the normal equation. State the equation of the resulting plane.

(b) Find the plane that instead minimizes the 1-norm of the residual. You may use the Minimize command in Mathematica.

(c) Compare and interpret the residual vectors $(Ax - b)$ for the solutions in (a) and (b).