Math 294 Exam 3 (Take-home)

Please answer each problem as clearly and completely as you can. Do not discuss these problems with other students, or anyone else but me. You may use your textbook, lecture notes, class materials (including those posted on the website), and homework, but do not use other books, the internet, or any materials other than those directly associated with the course. Please do feel free to ask me questions, either via email or coming by my office. Show all work to demonstrate that you understand your answer. You may use computational aids like Mathematica and calculators (indicate how you did so in your write-up).

This exam is due **4pm Wed May 10**. Late submissions will be penalized 10 points per day. There are 5 problems with a total of 100 points.

Problem 1 (20pt) In classical Markowitz portfolio optimization, we want to invest among a set of assets to form a portfolio with a desired balance between the risk and return. Let's consider a simple situation with 2 assets that have the same expected return but different volatility, where we invest a proportion x of our capital in asset 1 and the remainder $1-x$ in asset 2, letting x be any real number (negative numbers are allowed here). Suppose our only goal is to minimize the risk, as the portfolio's expected return here will be same for any x. If the standard deviation for asset i is σ_i, and the correlation between the assets' returns is ρ, then the covariance matrix for the two assets is

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

We seek the lowest risk portfolio, which mathematically is stated as follows:

$$\min_{x\in\mathbb{R}} \left[\begin{smallmatrix} x \\ 1-x \end{smallmatrix}\right]^T \Sigma \left[\begin{smallmatrix} x \\ 1-x \end{smallmatrix}\right].$$

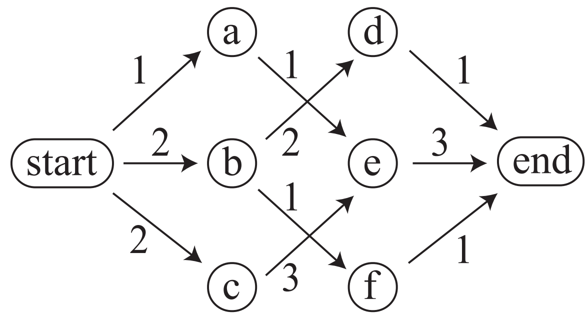
What type of optimization problem is this? Derive BY HAND the optimal value for x and also state $1-x$ in simplified form (as rational functions of the various parameters).

Problem 2 (20pt) The ellipse centered at point $c = (-4, 0)$ with shape matrix

$$P = \begin{bmatrix} 26 & -10 \\ -10 & 26 \end{bmatrix}$$

is defined by $\mathcal{E} = \{x : (x-c)^T P^{-1}(x-c) \le 1\}$. Determine the lengths and directions of the semi-axes of this ellipse and use this information to directly sketch the ellipse. Express this ellipse in the alternative form $\mathcal{E} = \{x = \hat{x} + Ru : \|u\|_2 \le 1\}$, again using the information about the semi-axes.

Problem 3 (20pt) Given the following capacity information about a pipe network, set up a convex optimization problem in standard form that seeks the maximal flow, then determine the optimal flow through each pipe and the total flow achieved through the network.



Problem 4 (20pt) Given the following table of the cost for each agent to do each task, set up a convex optimization problem in standard form that seeks the matching of agents to tasks (one agent per task and one task per agent) that minimizes the total cost, then determine an optimal matching and its cost.

Agent	Task 1	Task 2	Task 3	Task 4	Task 5	Task 6
#1	3	3	2	2	1	1
#2	2	3	3	4	2	3
#3	1	3	4	2	3	2
#4	3	2	2	1	3	2
#5	4	4	4	2	1	2
#6	1	4	2	3	4	4

Problem 5 (20pt) Solve the set of KKT conditions BY HAND, showing all steps, for the optimization problem

$$p^* = \min_{x \in \mathbb{R}^n} \sum_{j=1}^{n} e^{c_j x_j} \quad \text{s. t.} \quad \sum_{j=1}^{n} a_j x_j \geq b,$$

where a_j, c_j, and b are positive constants for $j = 1, \ldots, n$. Simplify as much as possible, stating the optimal x_j^* and the optimal value p^*. Explain why this problem is a convex optimization problem.