Math 294 Exercises on Basic Convex Problems

1. Chebyshev center of a square:

   (a) Find four inequalities of the form $a^T x \le b$ that yield the square $[0,1]\times[0,1]$ in $\mathbb{R}^2$. Sketch the square showing the four normal vectors for the sides.

   (b) Set up the standard form of a linear program for the problem of determining the center and radius of the largest disk that lies inside this square.

   (c) Solve this linear program by inspection (the answer is clear by symmetry).

2. Consider the linear program $p^* = \min_{x\in\mathbb{R}^2} f_0(x)$ subject to the constraints $x_1 + x_2 \le 1$, $4x_1 + 2x_2 \le 3$, $x_1 \ge 0$, and $x_2 \ge 0$.

   (a) Sketch the feasible set.

   (b) Find p^* and the optimal set if $f_0(x) = x_1 + x_2$. (Note: the optimal set can be empty, a single point, or more than one point.)

   (c) Find p^* and the optimal set if $f_0(x) = -x_1 - x_2$.

   (d) Find p^* and the optimal set if $f_0(x) = x_1 - x_2$.

   (e) Find p^* and the optimal set if $f_0(x) = x_2$.

   (f) Find p^* and the optimal set if $f_0(x) = \max\{x_1, x_2\}$.

3. Use a logarithm to transform the following geometric program into a linear program (LP):
$$p^* = \min_{x\in\mathbb{R}^3} x_1^{-1}\, x_2\, x_3$$
   subject to $x_1^{-1} \le 1$, $x_1 \le 3$, $x_1^2\, x_2^{-1/2}\, x_3 \le 2$, $x_1 > 0$, $x_2 > 0$, and $x_3 > 0$.

4. Use calculus to solve the following convex optimization problem:
$$p^* = \min_{x\in\mathbb{R}^2}(-2x_1 + x_2) \quad \text{s.t.} \quad x_1^2 + x_2^2 \le 5.$$
   (Identify critical points in the interior of the feasible region and minima on the boundary of the disk, then determine which of these yields the smallest value of the objective function.)

5. Express the least squares problem of finding the line that best fits the data points $\{(-2,-1),\ (0,0),\ (1,2),\ (3,5)\}$ as an unconstrained quadratic program (QP) with objective function in form $\frac{1}{2}x^T Hx + c^T x + d$. Specify the matrix H, vector c, and scalar d, where x is the vector whose components are the slope and y-intercept of the line.