Math 294 Exercises on Convex Sets

1. Sketch an example of a non-convex set and then draw its convex hull.

2. What is the distance between two parallel hyperplanes $\{x \in \mathbb{R}^n \,|\, a^T x = b_1\}$ and $\{x \in \mathbb{R}^n \,|\, a^T x = b_2\}$? Hint: use a projection onto the vector a.

3. Give conditions on the vectors $a, \tilde{a}$ and scalars $b, \tilde{b}$ such that

$$\{x \in \mathbb{R}^n \,|\, a^T x \leq b\} \subseteq \{x \in \mathbb{R}^n \,|\, \tilde{a}^T x \leq \tilde{b}\}.$$

4. Explain why each of the following sets is convex, by *directly* showing that the definition of convex set is satisfied.

   (a) A *slab* defined by $\{x \in \mathbb{R}^n \,|\, \alpha \leq a^T x \leq \beta\}$.

   (b) A *rectangle* defined by $\{x \in \mathbb{R}^n \,|\, \alpha_i \leq x_i \leq \beta_i,\ i = 1, \ldots, n\}$.

   (c) A *wedge* defined by $\{x \in \mathbb{R}^n \,|\, a_1^T x \leq b_1 \text{ and } a_2^T x \leq b_2\}$.

5. Prove that the sum $S_1 + S_2$ of two convex sets S_1 and S_2 in $\mathbb{R}^n$ is also convex, where

$$S_1 + S_2 = \{z_1 + z_2 \,|\, z_1 \in S_1, z_2 \in S_2\}.$$