# Math 294 Exercises on Duality

1. Consider the optimization problem

$$p^* = \min_{x \in \mathcal{D}} f_0(x) \;\text{ s.t. }\quad f_1(x) \le 0.$$

For each pair of functions $f_0$ and $f_1$ on the given domain $\mathcal{D}$, state whether the resulting problem is convex, state the dual problem, and then solve the primal and dual problems (determine $p^*$, $x^*$, $d^*$, and $\lambda^*$).

   (a) $f_0(x) = x$, $f_1(x) = x^2 - 1$, and $\mathcal{D} = \mathbb{R}$.

   (b) $f_0(x) = x^3$, $f_1(x) = 1 - x$, and $\mathcal{D} = \{x \in \mathbb{R} : x \ge 0\}$.

   (c) $f_0(x) = x$, $\mathcal{D} = \mathbb{R}$, and

$$f_1(x) = \begin{cases} -x + 2 & \text{if } x \ge 1 \\ x & \text{if } -1 \le x \le 1 \\ -x - 2 & \text{if } x \le -1 \end{cases}$$

2. Consider the optimization problem

$$p^* = \min_{x \in \mathbb{R}^2} (-x_1 x_2) \;\text{ s.t. }\quad x_1 + x_2^2 \le 2 \;\text{ and }\; x_1 \ge 0.$$

   (a) Sketch the feasible set and level sets of the objective function.

   (b) State the KKT conditions.

   (c) Solve the system of equations given by the KKT conditions. Be sure to check all possible cases of the two complementary slackness conditions.

3. Consider the optimization problem

$$p^* = \min_{x \in \mathbb{R}^2} \; \|x - x_0\|_2^2 \;\text{ s.t. }\quad \|x\|_2^2 \le 1.$$

where $x_0 = (-4, 3)$.

   (a) State the KKT conditions.

   (b) Solve the system of equations given by the KKT conditions.

   (c) Find the optimal value $p^*$ and optimal point $x^*$.

4. Consider the optimization problem

$$p^*(u) = \min_{x \in \mathbb{R}} \; x^2 - 4x \quad \text{s.t.} \quad x^2 - 1 \leq u$$

for some fixed number $u$ with $|u| < 1$.

(a) Determine $x_\lambda$ and $g(\lambda, u)$.

(b) Determine $\lambda_u$ and $d^*(u)$.

(c) Use the above to obtain $p^*(u)$ and $x^*(u)$.

(d) Verify that $\nabla_\lambda \, g(\lambda, u) = F(x_\lambda, u)$, where $F(x, u)$ is the vector formed by the constraint functions.