Math 294 Exercises on Lagrange Duality

1. Consider the simple optimization problem $p^* = \min_{x\in\mathbb{R}}(x^2+1)$ subject to the constraint $(x-2)(x-4) \leq 0$.

   (a) State the feasible set.

   (b) Find p^* and x^*.

   (c) Write out the Lagrangian function $\mathcal{L}(x,\lambda)$ and determine $\min_{x\in\mathbb{R}} \mathcal{L}(x,\lambda)$ in terms of λ. Verify that $p^* \geq \min_{x\in\mathbb{R}} \mathcal{L}(x,\lambda)$ for all $\lambda \geq 0$.

   (d) Plot the dual function $g(\lambda)$ for $\lambda \geq 0$, and then solve the dual problem

   $$d^* = \sup_{\lambda\geq 0} g(\lambda).$$

   State d^* and λ^*. Is d^* actually attained by the dual function?

   (e) Does strong duality hold for this problem?

2. Consider the piecewise-linear optimization problem $p^* = \min_{x\in\mathbb{R}} f_0(x)$, where $f_0(x) = \max\{|x+1|, |x-3|\}$, subject to the constraint $x \geq 0$.

   (a) Graph the objective function $f_0(x)$ and find p^* and x^*.

   (b) Write out the Lagrangian function $\mathcal{L}(x,\lambda)$ and graph it as a function of x for each of the fixed values $\lambda = \frac{1}{2}, 1, 2$.

   (c) Determine $g(\lambda) = \min_{x\in\mathbb{R}} \mathcal{L}(x,\lambda)$ in terms of λ (as a piecewise function).

   (d) Solve the dual problem $d^* = \sup_{\lambda\geq 0} g(\lambda)$ and state d^* and λ^*.

   (e) Does strong duality hold for this problem?

3. Economic interpretation of the dual problem: Suppose a small shop makes wooden toys, where each toy train requires one piece of wood and 2 tins of paint, while each toy boat requires one piece of wood and 1 tin of paint. The profit on each toy train is \$30, and the profit on each toy boat is \$20. Given an inventory of 80 pieces of wood and 100 tins of paint, how many of each toy should be made to maximize the profit?

   (a) Write out the optimization problem in standard form, writing all constraints as inequalities.

   (b) Sketch the feasible set and determine p^* and x^*.

(c) Find the dual problem, then determine d^* and λ^*. Note that we can interpret the Lagrange multipliers λ_k associated with the constraints on wood and paint as the prices for each piece of wood and tin of paint, so that $-d^*$ is how much money would be obtained from selling the inventory for those prices. Strong duality says a buyer should not pay more for the inventory than what the toy store would make by producing and selling toys from it, and that the toy store should not sell the inventory for less than that.

(d) The other interpretation of the Lagrange multipliers is as sensitivities to changes in the constraints. Suppose the toymaker found some more pieces of wood; the λ_k associated with the wood constraint will equal the partial derivative of $-p^*$ with respect to how much more wood became available. Suppose the inventory increases by one piece of wood. Use λ^* to estimate how much the profit would increase, without solving the updated optimization problem. How is this consistent with the price interpretation given above for the Lagrange multipliers?