Math 294 Exercises on Quadratic Problems

1. Consider the optimization problem

$$p^* = \min_{x \in \mathbb{R}^4} \left(\frac{1}{2} x^T H x + c^T x \right) \quad \text{s.t.} \quad Ax = b,$$

where

$$H = \begin{bmatrix} 5 & -1 & 0 & 0 \\ -1 & 5 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & -1 & 0 & 4 \\ 1 & 1 & -2 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Solve this QP by first finding the general solution to the linear system $Ax = b$, substituting it into the quadratic objective function, and then stating and solving the resulting unconstrained quadratic program. State z^*, x^*, and p^*.

2. Draw careful sketches of the zero-level sets of the the quadratic inequality $\frac{1}{2} x^T H x + c^T x + d \leq 0$ for the following cases:

   (a) $H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $d = -2$

   (b) $H = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $d = -2$

   (c) $H = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $d = -1$

   (d) $H = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $d = -1$

3. Suppose we want to fit a line to the set of points $\{(-1, -4), (0, 1), (1, 3), (2, 4), (3, 8)\}$.

   (a) Solve the least squares problem that minimizes the 2-norm of the residual (satisfies the normal equation).

   (b) Find the line that minimizes the 1-norm of the residual. How does this solution differ qualitatively from the least squares solution?

   (c) Solve the constrained least squares problem of finding the line that minimizes the 2-norm of the residual subject to the y-intercept equaling zero. Compare the resulting slope with the slope found in part (b).