Fall 2018 STAT 360 Exam 2 Review Problems

The exam will be in class on Tuesday, December 4, and will cover Chapters 6-8 and 10.2. You may use one sheet of notes (front and back of an 8.5x11in piece of paper), which is to be turned in with the exam. You may use a calculator during the exam; you may not use a phone or borrow a calculator from a classmate during the exam. Please turn off all phones and anything else that might beep or be a distraction.

Practice exam problems

1. Suppose X and Y are jointly continuous random variables with joint pdf given by

$$f(x, y) = x + y \text{ for } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \text{ (and 0 otherwise).}$$

   a. Set up a computation to compute P(X > 3Y). Do not evaluate.
   b. What is the marginal pdf of X?
   c. Find the conditional pdf for Y given X.
   d. Find the expected value of Y given X.
   e. Use the Law of Total Expectation to calculate E[Y].

2. The proportion of time per day that all checkout counters in a supermarket are busy is modeled as a random variable Y with density given by $f(y)=cy^2(1-y)^4$ for $0<y<1$, and 0 otherwise.
   a. What value of *c* makes this a valid pdf?
   b. What distribution does Y have? (Be specific giving the distribution name and parameters.)
   c. What is the expected proportion of time per day that all counters are busy?

3. Starting at 9 am, students arrive at a classroom at a rate of 2 per minute. There are 30 students in the class, and the class starts at exactly 9:15 am.
   a. What is the expectation and variance of the number of students in class by 9:15 am?
   b. To find the probability that the last student who arrives is late, what distribution would you use?
   c. What is the expected time of arrival of the seventh student who gets to class?

4. Let X and Y be jointly distributed continuous random variables with joint pdf given by

$$f(x,y) = ½ \text{ for } x > 0,\ y > 0 \text{ and } x + y < 2, \text{ and 0 otherwise.}$$

   a. Find the marginal of X.
   b. Find E[Y].
   c. Find P(X > 2Y).
   d. Find P(Y > 1| X = 0.5).
   e. Find E[Y | X].
   f. Find V[Y | X].

5. Let X be a continuously distributed random variable with pdf $f(x) = kx^2$ for $0<x<2$, and 0 otherwise.
   a. Find the value of *k* that makes this a valid pdf.
   b. Find P(X > 1).
   c. Find the expected value of $X^3 - 1$.

6. Suppose X has the pdf $f(x) = \frac{1}{36} x e^{-x/6}$ for $x > 0$, and 0 otherwise. Without doing any integration, determine E[X] and E[X^2], and identify the distribution of X, including all parameter values.

7. Let X be uniformly distributed over the interval from 0 to $\pi/2$. Find the density function of $Y=\sin(X)$. You may use the fact that $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$.

8. Suppose that two components have independent exponentially distributed lifetimes T_1 and T_2 with parameters α and β, respectively. Find $P(T_1 > T_2)$.

9. A card contains 3 chips and has an error-correcting mechanism such that the card still functions if a single chip fails but does not function if two or more chips fail. If each chip has a lifetime that is an independent exponential with parameter $\lambda = 2$, find the density function of the card's lifetime.

10. Let U_1, U_2 be independent Uniform(0,1) random variables.
    a. Find the pdf of $\min(U_1, U_2)$. What distribution is this?
    b. Find the pdf of $\max(U_1, U_2)$. What distribution is this?

11. The lifetime of a transistor radio is T years, where T is an exponential variable with parameter $\lambda = .5$. What is the probability that the radio will last at least 5 years? Given that is lasts 10 years, what is the probability that it will last at least 5 more years beyond that?

12. Let X be uniform on [0,1]. Find the density function of $Y=X^3$ and use to compute $E[X^3]$. Compare to directly computing $E[X^3]$ using the density function of X.

13. Let X and Y have joint density $f_{X,Y}(x,y)=2e^{-x-y}$ for $0 < x < y < \infty$.
    a. Are X and Y independent? Justify using marginal densities.
    b. Find $P(X+Y < 1)$.

14. Let X and Y be independent normal random variables with means 2 and 5 respectively, and standard deviations 3 and 4, respectively. What is the distribution of $W = X + Y$? Specify all parameter values.

15. If you want the most precise measurement of the weight of a very light object, would it be better to weigh it one time on a digital scale which reports the weight with an error which has a standard deviation of 0.037g around the true weight or weigh it seven times on a balance which reports the true weight of the object with an error with a standard deviation of 0.094g? Be sure to justify your response.

16. Let X be uniformly distributed on the interval [1, 2]. Find E[X] and E[1/X]. Is E[1/X] = 1/E[X]?

17. Name that Distribution or Process
    a. Has the memoryless property (discrete or continuous)
    b. Originally used to model income distribution in a country
    c. Generalization of binomial
    d. Distribution with 68-95-99.7 rule
    e. If you turned to a random page in your textbook and looked at the first digit on the page (ignoring 0s), how to determine how likely seeing any particular digit would be
    f. Useful for modeling the difference in successive arrival times

g. When computing an average, if you start with iid random variables from this distribution, their average also has this distribution (with different parameter values)
h. Counts numbers of arrivals in some interval of time
i. Can be used to model cumulative time till arrival of the nth event in a sequence of events

18. Let X be a random variable with pdf $f(x) = kx^4$, for $0 < x < 1$, and 0 otherwise.
   a. Determine the value of k that makes this a valid pdf.
   b. Identify the distribution of X, specifying all parameters.
   c. What is E[X]?
   d. Find the pdf for $Y = X^2$.

19. Find the density function of Z = X + Y when X and Y have joint density function

$$f(x,y) = \frac{1}{2}(x+y)e^{-(x+y)} \text{ for } x \geq 0 \text{ and } y \geq 0, \text{ and 0 otherwise.}$$

20. Suppose X and Y are independent RVs with X~Gamma(a,λ) and Y~Gamma(b, λ).
   a. Compute the joint density of U=X+Y and V=X/(X+Y).
   b. Show that U and V are independent.
   c. Identify the distributions for U and V, specifying all parameters.

Brief solutions so you can check that you are on the right track. Feel free to swing by office hours to go over any of the problems on the review. Please let me know if you find any errors!

1. a) $\int_0^1 \int_0^{x/3} (x+y)\,dy\,dx$

   b) $f_X(x) = x+1/2$ for $0 \leq x \leq 1$

   c) $f_{Y|X}(y|x) = \frac{x+y}{x+1/2}$ for $0 \leq x, y \leq 1$

   d) $E[Y|X] = \frac{3x+2}{6x+3}$

   e) E[Y]=E[E[Y|X]] = 7/12

2. Y~Beta(3,5), c=105, and E[Y] = 3/8.
3. a) Treat as Poisson process, N_t ~ Pois($2t$), t in minutes. Expectation and variance are both 30.
   b) The arrival time of the last student (cumulative from time 0) is given by S_{30}~Gamma(30,2) The student will be late if the time has advanced past 9:15, so we want $P(S_{30}>15)$.
   c) $E[S_7]$=7/2 because S_7~Gamma(7,2), so 9:03:30am.
4. a) $f_X(x)=1-x/2$ for $0<x<2$

   b) E[Y]=2/3   c) 1/3   d) 1/3   e) E[Y|X]=1-X/2   f) $V[Y|X]=(2-X)^2/12$

5. a) k=3/8

   b) P(X > 1) = 7/8

   c) $E[X^3 - 1] = 3$

6. X~Gamma(2,1/6) so E[X]=12, V[X]=72, and $E[X^2]=V[X]+E[X]^2=216$.
7. First find cdf then take derivative to get pdf: $f_Y(y)=\frac{2}{\pi\sqrt{1-y^2}}$ for $0 \leq y \leq 1$.
8. $P(T_1 > T_2) = \int_0^\infty \int_0^{t_1} \alpha\beta e^{-\alpha t_1 - \beta t_2}\,dt_2\,dt_1 = \frac{\beta}{\alpha+\beta}$
9. $f(y) = 12e^{-4y}(1-e^{-2y})$ for $y>0$ (find cdf first, then take derivative to get pdf)
10. Beta(1,2) and Beta(2,1)

11. Answer is same for both questions by memoryless property, $e^{-2.5} \approx 0.082$.
12. $f_Y(y)=(1/3)y^{-2/3}$ for $0 \le y \le 1$ and $E[X^3]=1/4$.
13. Not independent, product of marginal does not equal the joint pdf. $f_X(x)=2e^{-2x}$ and $f_Y(y)=2e^{-y}(1-e^{-y})$. $P(X+Y < 1)=1-2/e$ (sketch the region and take care in setting up the double integral here).
14. Normally distributed with mean 7 and variance 25.
15. Assuming the weighings are independent, the variance for the first procedure is $0.037^2=0.001369$ and the variance for the second procedure is $0.094^2/7=0.001262$, so the second procedure is a little better (more likely to have smaller error).
16. Not equal, 2/3 vs 0.693.
17. a) Geometric and exponential; b) Pareto; c) Multinomial; d) Normal; e) Benford's law; f) Exponential; g) Normal; h) Poisson; i) Gamma.
18. a) k=5   b) X~Beta(5,1)   c) $E[X]=a/(a+b)=5/6$

    d) $f_Y(y)=(5/2)y^{3/2}$ for $0 < y < 1$ and 0 otherwise.

19. A good approach is to let Z=X+Y and T=Y, so X=Z-T and Y=T. Jacobian J=1. Conversion formula from 10.2 gives $f_{Z,T}(z,t)= \frac{1}{2} ze^{-z}$. Integrate out the t to get $f_Z(z)= \frac{1}{2} z^2 e^{-z}$ for $z \ge 0$.
20. a) Solve for x and y to obtain $x=uv$ and $y=u(1-v)$, find the Jacobian, then plug into the conversion formula from 10.2 and simplify. Note that the joint density for X and Y will be the product of their Gamma pdfs because the RVs are independent.

    b) Joint pdf for U and V can be written as the product of the marginal density for U and the marginal density for V, so they are independent.

    c) U~Gamma(a+b, λ) and V~Beta(a,b)