STAT 360 Probability Exam 2
December 4, 2018

Name: Solutions

1. Show all work and explain your method to receive full credit. You may receive partial credit for partially completed problems. *Be sure to always specify the domain when finding functions like a density.* Please simplify fractions and other values when possible.

2. You may use calculators and a two-sided sheet of notes. You may not use any other references, texts, or other aids. You may NOT use your phone during the exam (except in an emergency).

3. You may not discuss the exam with anyone but Prof Leise. **Uphold the honor code.**

4. Suggestion: Begin by reading all of the questions and then work first on problems you understand the best.

5. Staple your sheet of notes and any scratch paper with work you want me to see (clearly labeled) to your exam before turning it in.

6. Initial in the box below to acknowledge that you have read and understand the instructions.

Problem #	1	2	3	4	5	Total
Points earned						
Possible points	20	20	20	20	20	100

1. Suppose $U \sim \text{Unif}(0,1)$. Define the random variable $Y = \frac{1}{1+U}$.

   a. Calculate the probability that $Y > 3/4$.

   b. Find the probability density function for Y. Be sure to specify the domain.

   c. Calculate E[Y].

$$P(Y > 3/4) = P\left(\frac{1}{1+U} > 3/4\right) = P(4/3 > 1+U)$$
$$= P(U < 1/3)$$
$$= 1/3$$

$$F_Y(y) = P(Y<y) = P\left(\frac{1}{1+U} < y\right) = P\left(\frac{1}{y} < 1+U\right)$$
$$= P\left(U > \frac{1}{y} - 1\right)$$
$$= 1 - F_U\left(\frac{1}{y} - 1\right)$$

$$f_Y(y) = F_Y'(y) = -F_U'\left(\frac{1}{y}-1\right) \cdot \frac{d}{dy}\left(\frac{1}{y}-1\right)$$
$$= -f_U\left(\frac{1}{y}-1\right) \cdot \left(-\frac{1}{y^2}\right)$$
$$= (-1)\left(-\frac{1}{y^2}\right)$$
$$= \frac{1}{y^2} \text{ for } \frac{1}{2} < y < 1$$

(plug $u=0$ and $u=1$ into $y=\frac{1}{1+u}$ to get y-domain)

$$E[Y] = \int_{1/2}^{1} y \cdot \frac{1}{y^2}\, dy = \int_{1/2}^{1} \frac{1}{y}\, dy = \cancel{\ln 1}^{0} - \ln \tfrac{1}{2}$$
$$= \boxed{\ln 2}$$

OR

$$E[Y] = E\left[\frac{1}{1+U}\right] = \int_0^1 \frac{1}{1+u} \cdot 1\, du = \ln(1+u)\Big|_0^1 = \ln 2$$

2. Suppose X is a uniformly distributed RV on [1,2], and given X=x, Y is exponentially distributed with λ=x. That is, X ~ Unif(1,2) and Y|X=x ~ Exp(x).
   a. Calculate E[Y].
   b. Calculate V[Y].

$$E[Y] = E[E[\underbrace{Y|X}_{\sim \mathrm{Exp}(X)}]] = E\left[\frac{1}{X}\right] = \int_1^2 \frac{1}{x} \cdot \frac{1}{2-1}\,dx$$

$$= \ln 2$$

$$V[Y] = E[V[Y|X]] + V[E[Y|X]]$$

$$= E\left[\frac{1}{X^2}\right] + V\left[\frac{1}{X}\right]$$

$$= E\left[\frac{1}{X^2}\right] + E\left[\frac{1}{X^2}\right] - \left(E\left[\frac{1}{X}\right]\right)^2$$

$$= 2E\left[\frac{1}{X^2}\right] - (\ln 2)^2$$

$$= 2\int_1^2 \frac{1}{x^2}\,dx - (\ln 2)^2$$

$$= 2\left(-\frac{1}{x}\Big|_1^2\right) - (\ln 2)^2$$

$$= 1 - (\ln 2)^2$$

3. Let X and Y be jointly distributed continuous RVs whose joint pdf is given by

$$f(x, y) = \tfrac{1}{3}(x + y) \text{ for } 0 \le x \le 2 \text{ and } 0 \le y \le 1 \text{ (and 0 otherwise).}$$

a. Find the marginal distribution of X.

b. Find the conditional distribution of Y given that X = x.

c. Find E[Y | X = x].

d. Use the Law of Total Expectation to calculate E[Y].

$$f_X(x) = \int_0^1 \tfrac{1}{3}(x+y)\,dy = \tfrac{1}{3}xy + \tfrac{1}{6}y^2 \Big|_{y=0}^{y=1} = \tfrac{1}{3}x + \tfrac{1}{6} \quad \text{for } 0 \le x \le 2$$

$$f_{Y|X}(y|x) = \frac{\frac{1}{3}(x+y)}{\frac{1}{3}x + \frac{1}{6}} = \frac{x+y}{x+1/2} \quad \text{for } 0 \le x \le 2,\ 0 \le y \le 1$$

$$E[Y|X=x] = \int_0^1 y \cdot \frac{x+y}{x+1/2}\,dy = \frac{\frac{1}{2}xy^2 + \frac{1}{3}y^3 \Big|_{y=0}^{y=1}}{x+1/2}$$

$$= \frac{\frac{1}{2}x + \frac{1}{3}}{x+1/2} = \frac{3x+2}{6x+3} \quad \text{for } 0 \le x \le 2$$

$$E[Y] = E[E[Y|X]] = E\left[\frac{3X+2}{6X+3}\right] = \int_0^2 \frac{3x+2}{6(x+\frac{1}{2})} \cdot \tfrac{1}{3}(x+\tfrac{1}{2})\,dx$$

$$= \int_0^2 \tfrac{1}{6}x + \tfrac{1}{9}\,dx$$

$$= \tfrac{1}{12}x^2 + \tfrac{1}{9}x \Big|_0^2 = 5/9$$

4. Starting at 8:50am, students arrive at a classroom at an average rate of 2 students per minute, following a Poisson process. There are 24 students in the class, and the class starts at exactly 9:00am.

    a. What is the expected value and standard deviation for the number of students in the classroom when class starts at 9:00am?

    b. Write an expression to calculate the probability that exactly 22 students are in class at 9:00am.

    c. To find the probability that the last student to arrive is late, what continuous distribution could you use? Specify the parameters.

    d. What is the expected time of arrival of the last student to arrive at the classroom?

$E[N_{10}] = 2 \cdot 10 = 20$ students

$V[N_{10}] = 20$ so $SD[N_{10}] = \sqrt{20} \approx 4.5$ students

(typical range will be 16-24 students in class at 9am)

$$P(N_{10} = 22) = \frac{e^{-20} 20^{22}}{22!} \approx 0.077$$

$P(S_{24} > 10)$ uses Gamma(24, 2) distribution

$$E[S_{24}] = \frac{24}{2} = 12 \text{ min so } 9{:}02\text{am}$$

5. Suppose X and Y are independent RVs with X ~ Gamma(a, λ) and Y ~ Gamma(b, λ).

   a. Compute the joint density of U=X+Y and V=X/(X+Y). Be sure to indicate the values U and V can take.

   b. Show that U and V are independent.

   c. Identify the distributions for U and V, specifying all parameters.

Solve $u = x+y$, $v = \frac{x}{x+y}$ $\Rightarrow$ $x = uv$, $y = u(1-v)$

Domain: $x>0$ and $y>0$

so $u = x+y > 0$

$v = \frac{x}{x+y} > 0$

and $x < x+y \Rightarrow \frac{x}{x+y} < 1$

Hence $u>0$ and $0<v<1$

Jacobian $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ (1-v) & -u \end{vmatrix} = -u$

$$f_{U,V}(u,v) = f_{X,Y}(uv, u(1-v)) \cdot |-u|$$

$$= \frac{\lambda^a (uv)^{a-1} \cancel{e^{-\lambda uv}}}{\Gamma(a)} \cdot \frac{\lambda^b u^{b-1} (1-v)^{b-1} e^{-\lambda u} \cancel{e^{\lambda uv}}}{\Gamma(b)}$$

$$= \lambda^{a+b} u^{a+b-1} e^{-\lambda u} \cdot \frac{1}{\Gamma(a)\Gamma(b)} v^{a-1}(1-v)^{b-1}$$

rearrange as function of u times function of v

$$= \underbrace{\frac{\lambda^{a+b} u^{a+b-1} e^{-\lambda u}}{\Gamma(a+b)}}_{\substack{f_U(u) \\ \text{for } u>0}} \cdot \underbrace{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} v^{a-1}(1-v)^{b-1}}_{\substack{f_V(v) \\ \text{for } 0<v<1}}$$

multiply & divide by $\Gamma(a+b)$ so matches distribution fns for Gamma & Beta RVs

U and V are independent because joint pdf is separable into function of u times function of v

Looking at $f_U(u)$, we see $U \sim \text{Gamma}(a+b, \lambda)$

$f_V(v)$ $\qquad V \sim \text{Beta}(a, b)$